We want to figure out how to define

$$\int f \, d\mu$$

$(\Omega, \mathcal{F}, \mu)$ measure space

$f : \Omega \to \mathbb{R}$ Borel measurable.

Special case

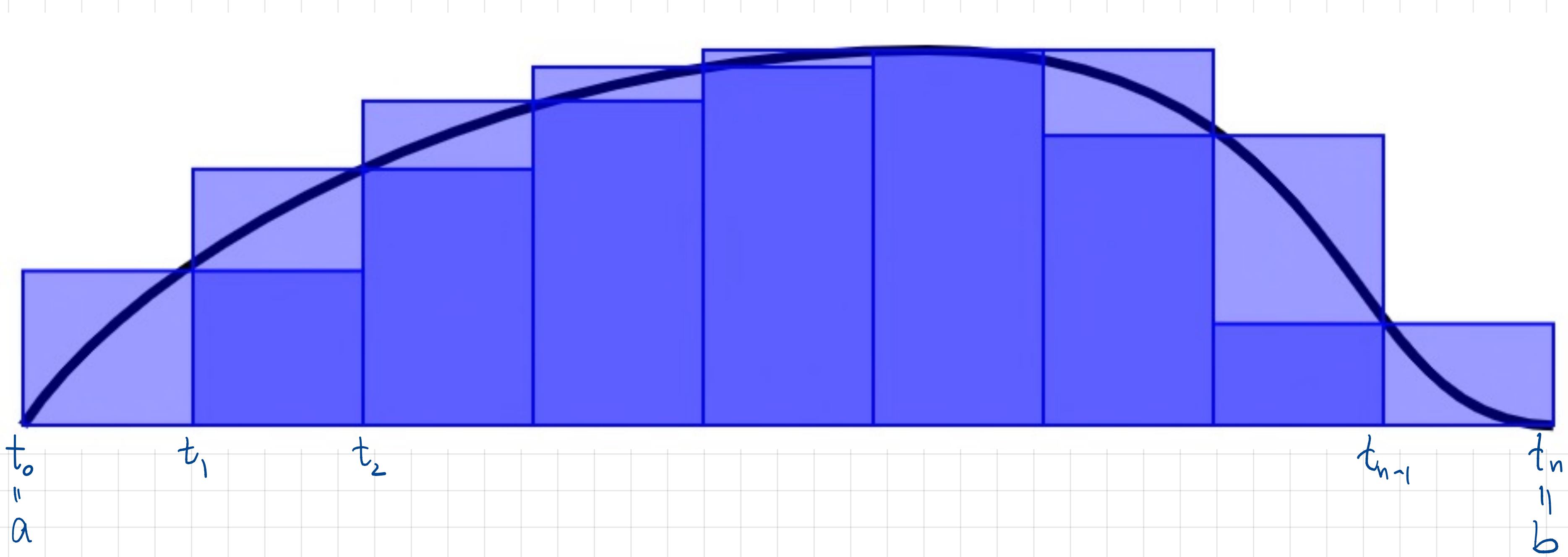
Stieltjes Radon measure determined by

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$

$$\mu_F (a,b] = F(b) - F(a)$$

## Riemann-Stieltjes Integral [Driver, §11.1]

Define $\int_a^b f \, d\mu_F = \int_a^b f(x) \, dF(x)$ as follows:




Consider all partitions

$$\pi = \{ a = t_0 < t_1 < \cdots < t_n = b \}$$

$$\overline{S}_\pi := \sum_n \sup_{t_{n-1} \leq t \leq t_n} f(t) \cdot \mu_F (t_{n-1}, t_n]$$

$$\underline{S}_\pi := \sum_n \inf_{t_{n-1} \leq t \leq t_n} f(t) \cdot \mu_F (t_{n-1}, t_n]$$

$$\overline{\int} f = \inf_\pi \overline{S}_\pi \qquad \underline{\int} f = \sup_\pi \underline{S}_\pi$$

$\int f$ if $=$

## Defects:

* Only works on a compact interval
* Only works for bounded functions
* Is not robust under many limits (only finitely additive)
* Fails for many simple functions.

Eg. $f = \mathbb{1}_{\mathbb{Q}}$ on $[a,b]$

## What's the problem?

The Riemann-Stieltjes integral is designed for functions well-adapted to an interval partition of the domain. I.e. works best if

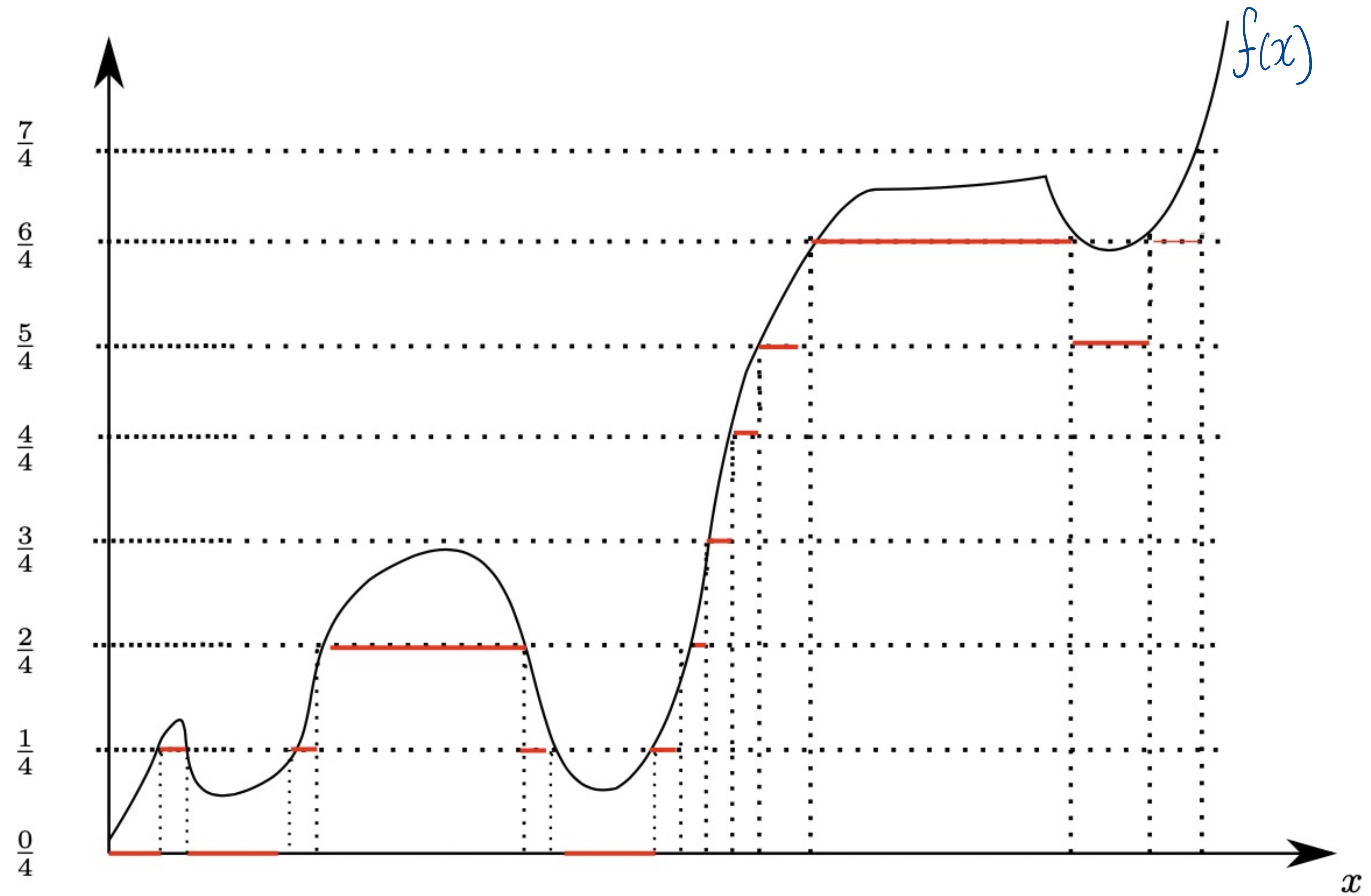
$$f^{-1}(s,t) \text{ is a nice union of intervals } \forall s<t$$

(like continuous functions.)

**Theorem:** [11.5] A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable ( $F(x)=x$, i.e. $\mu_F=\lambda$ Lebesgue measure)

iff $\{x\in[a,b] : f \text{ is discontinuous at } x\}$

# What's the Fix?




Partition the **range**, not the domain.

→ The resulting approximation will be flat – not necessarily on intervals, but on measurable sets. And we know how to measure those!