We want to figure out how to define

$$\int f \, d\mu \qquad (\Omega, \mathcal{F}, \mu) \text{ measure space}$$

$f : \Omega \to \mathbb{R}$ Borel measurable.

Special case

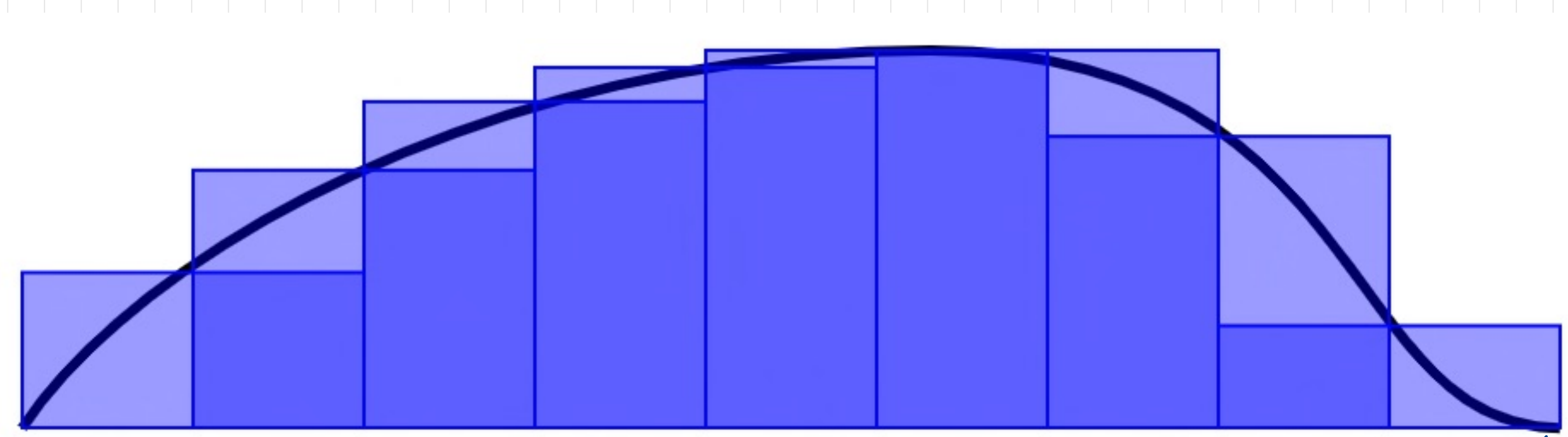
$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$

Stieltjes Radon measure determined by

$$\mu_F(a,b] = F(b) - F(a)$$

Riemann-Stieltjes Integral [Driver, §11.1]

Define $\displaystyle\int_a^b f \, d\mu_F = \int_a^b f(x)\, dF(x)$ as follows:



Consider all partitions

$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

$$\overline{S}_\pi := \sum_n \sup_{t_{n-1} \le t \le t_n} f(t) \cdot \mu_F(t_{n-1}, t_n]$$

$$\underline{S}_\pi := \sum_n \inf_{t_{n-1} \le t \le t_n} f(t) \cdot \mu_F(t_{n-1}, t_n]$$

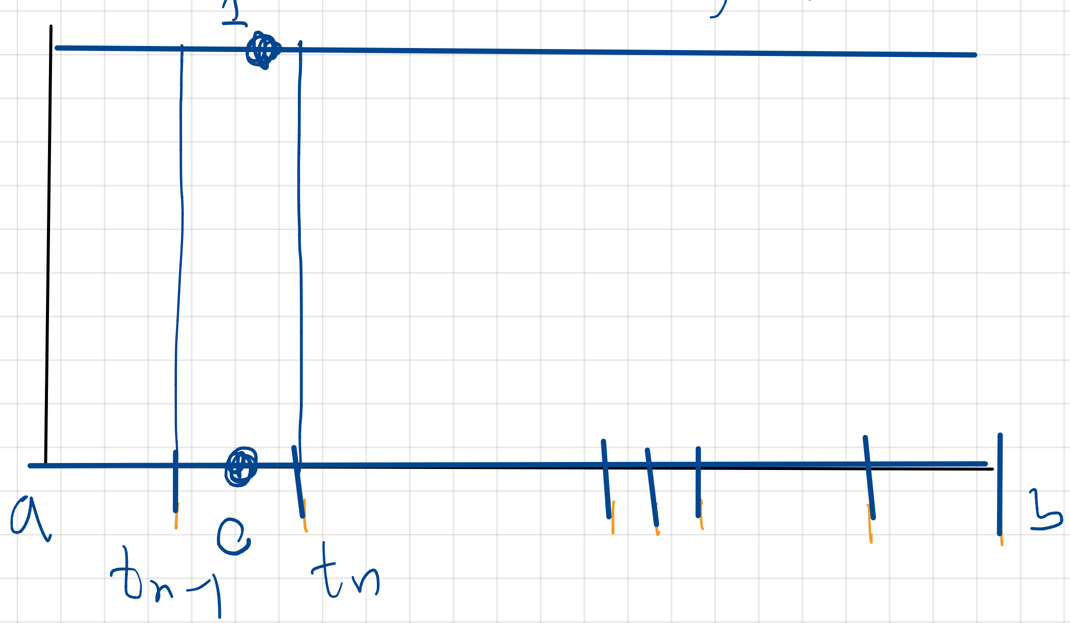
$$\overline{\int} f = \inf_\pi \overline{S}_\pi \qquad \underline{\int} f = \sup_\pi \underline{S}_\pi$$

$$\int f \quad \text{if} \quad =$$

Defects:

* Only works on a compact interval
* Only works for bounded functions
* Is not robust under many limits (only finitely additive)
* Fails for many simple functions.

Eg. $f = \mathbb{1}_{\mathbb{Q}}$ on $[a,b]$



$$\sup_{[s,t]} f = 1 \qquad \inf_{[s,t]} f = 0.$$

$$\overline{S}_\pi = \sum_n 1 \cdot [F(t_n) - F(t_{n-1})] = F(b) - F(a)$$

$$\underline{S}_\pi = \sum_n 0 \cdot (\text{"} \quad \text{"}) = 0.$$

$\mathbb{1}_{\mathbb{Q}}$ is RS-integrable on $[a,b]$ iff $F \equiv$ const. on $[a,b]$.

What's the problem?

The Riemann-Stieltjes integral is designed for functions well-adapted to an interval partition of the domain. I.e. works best if

$$f^{-1}(s,t) \text{ is a nice union of intervals } \forall s<t$$

(like continuous functions.) $\mathbb{1}_{\mathbb{Q}}$ oscillates fast on all scales.

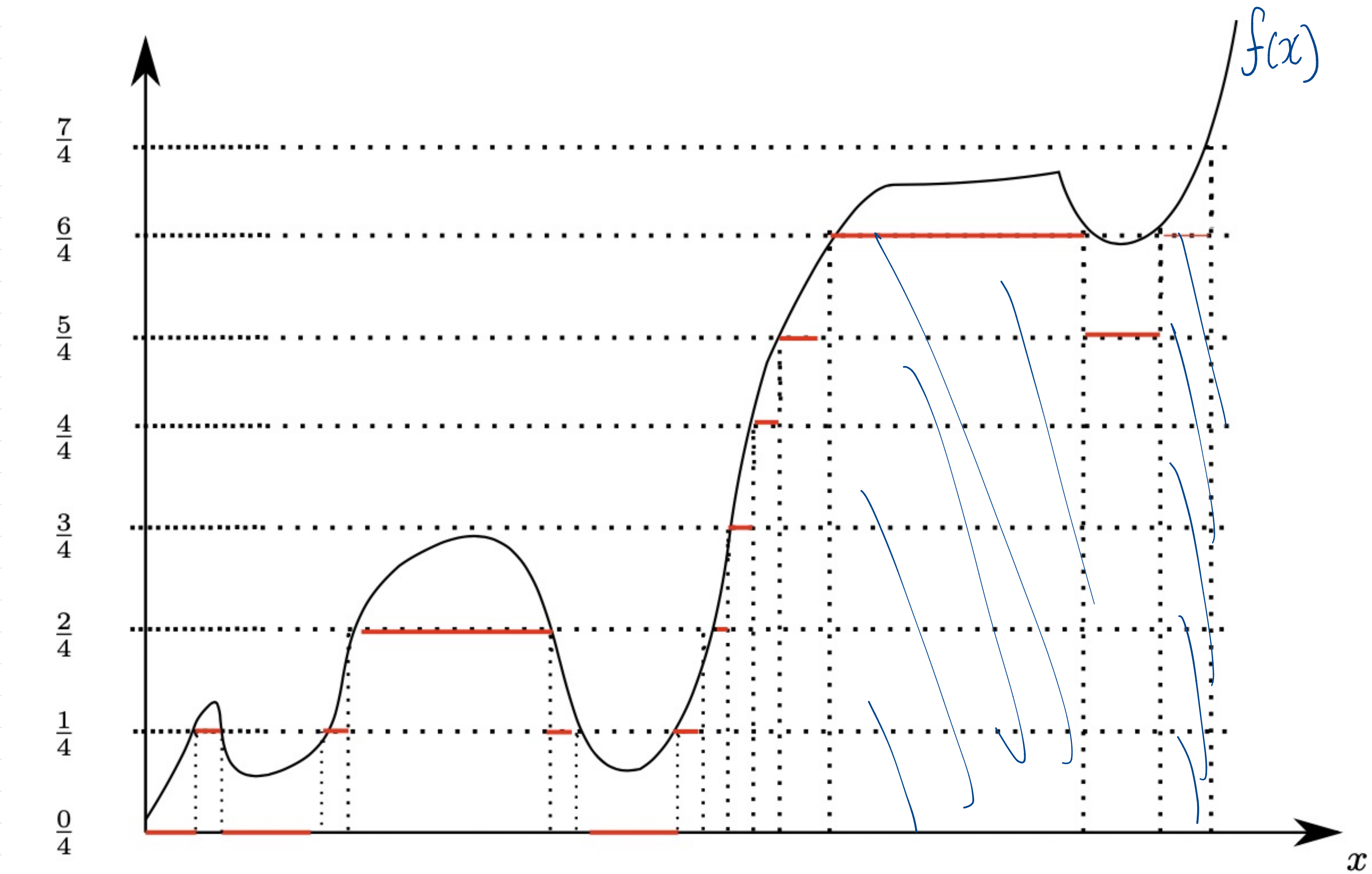
Theorem: [11.5] A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable ($F(x)=x$, i.e. $\mu_F=\lambda$ Lebesgue measure)

$$\text{iff} \quad \bar{\lambda}\{x\in[a,b] : f \text{ is discontinuous at } x\} = 0.$$

completion

need not be in $\mathcal{B}(\mathbb{R})$

What's the Fix?



$$\sum_n \; \sup \text{ or } \inf \; \mu\left(f^{-1}(t_{n-1}, t_n]\right)$$

$f^{-1}(t_{n-1}, t_n]$

t_n $\quad$ t_{n-1}

Partition the **range**, not the domain.

The resulting approximation will be flat – not necessarily on intervals, but on measurable sets. And we know how to measure those!