Robustness of Measurability [Driver §9.1–9.2]

We showed last time that if $f : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ measurable, then

$$\boxed{X_1, \dots, X_d \text{ are } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \Rightarrow f(X_1, \dots, X_d) \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable}}$$

In particular, applies to continuous $f : \mathbb{R}^d \to \mathbb{R}$.

Important Examples:

- $(x,y) \mapsto \max(x,y) = x \vee y$ and $(x,y) \mapsto \min(x,y) = x \wedge y$ are continuous.

$\therefore$ $X_+ := \max(X, 0)$ and $X_- := -\min(X, 0)$ are measurable.

If $f:\mathbb{R}^d \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, then

$$\boxed{X_1,\dots,X_d \text{ are } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \Rightarrow f(X_1,\dots,X_d) \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable}}$$

Often useful in dealing with some discontinuous functions of random variables.

Eg. The function $i_\alpha(x) = \begin{cases} 1/x, & x \neq 0 \\ \alpha, & x = 0 \end{cases}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable.
(for any $\alpha \in \mathbb{R}$)

$\therefore$ If $f:\Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, so is $i_\alpha \circ f$

Measurability and "Limits"

Proposition [9.14] If $f_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are measurable, then so are

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n .$$

Pf. $(\sup_n f_n)^{-1}(-\infty, t] = \{x \in \mathbb{R} : \sup_n f_n \le t\}$

$$\limsup_{n\to\infty} f_n = \inf_n \sup_{j\ge n} f_j$$

Simple Functions

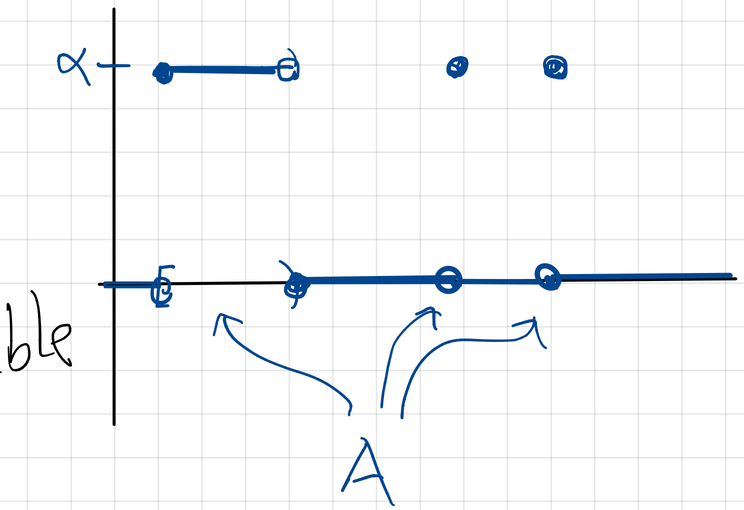
A function $\varphi : \Omega \to \mathbb{R}$ is called **simple** if $\varphi(\Omega)$ is a finite set.

E.g. $\varphi = \alpha \mathbb{1}_A$

↑

$\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

iff



Lemma: A simple function $\varphi : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable iff $\varphi^{-1}(\{\alpha\}) \in \mathcal{F}$ $\forall \alpha \in \mathbb{R}$.

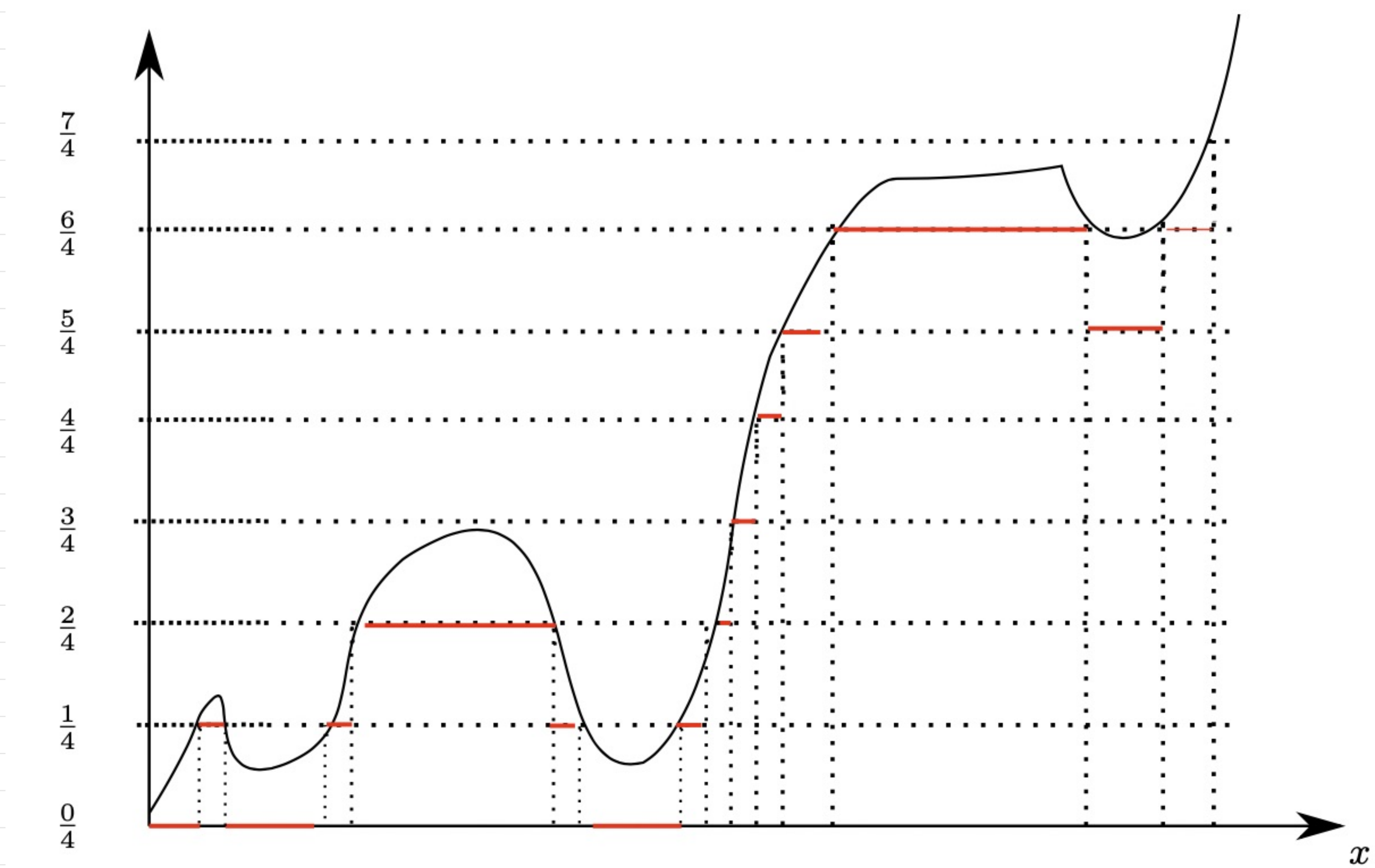
Simple Approximation

Theorem: [9.41] If $f:(\Omega,\mathcal{F})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is measurable, there is a sequence φ_n of simple measurable functions s.t.

$$\lim_{n\to\infty}\varphi_n(x)=f(x)\quad \forall x\in\Omega$$

Moreover, $\varphi_n\to f$ uniformly on $f^{-1}[-M,M]$ $\forall M>0$.

Pf. Suffices to assume $f\geq 0$



$$\varphi_n(x)=\sum_{k=0}^{2^{2n}-1}\frac{k}{2^n}\mathbb{1}_{\{\frac{k}{2^n}<f\leq\frac{k+1}{2^n}\}}(x)+2^n\mathbb{1}_{\{f>2^n\}}(x)$$

Factoring Random Variables (Doob-Dynkin Representation)

$$\left.\begin{array}{l} f:\mathbb{R}^d \to \mathbb{R} \text{ Borel measurable, and} \\ X_1,\dots,X_d \text{ are } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \end{array}\right\} \Rightarrow f(X_1,\dots,X_d) \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable.}$$

Cor. [9.43] $X_1,\dots,X_d$ $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

& $Y:\Omega\to\mathbb{R}$ $\sigma(X_1,\dots,X_d)/\mathcal{B}(\mathbb{R})$-measurable

$\Rightarrow \exists$ Borel measurable $f:\mathbb{R}^d\to\mathbb{R}$ s.t. $Y = f(X_1,\dots,X_d)$.

To prove this, we need another characterization of $\sigma(X_1,\dots,X_d)$.

Lemma: Let $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_d \end{bmatrix} : \Omega \to \mathbb{R}^d$. Then

$$\sigma(X_1,\dots,X_d) =$$

$\shortparallel$

smallest σ-field over Ω wrt which $X_1,\dots,X_d$ are measurable (with respect to $\mathcal{B}(\mathbb{R})$)

Pf. [HW].

Cor. [9.43] $X_1, \ldots, X_d$ $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

& $Y : \Omega \to \mathbb{R}$ $\sigma(X_1, \ldots, X_d)/\mathcal{B}(\mathbb{R})$-measurable

$\Rightarrow \exists$ Borel measurable $f : \mathbb{R}^d \to \mathbb{R}$ s.t. $Y = f(X_1, \ldots, X_d)$.

Pf. To begin, assume $Y = \mathbb{1}_A$