Robustness of Measurability [Driver §9.1–9.2]

We showed last time that if $f: \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ measurable, then

$$\boxed{X_1, \dots, X_d \text{ are } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \Rightarrow f(X_1, \dots, X_d) \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable}}$$

In particular, applies to continuous $f: \mathbb{R}^d \to \mathbb{R}$.

Important Examples:

- $(x,y) \mapsto \max(x,y) = x \vee y$ and $(x,y) \mapsto \min(x,y) = x \wedge y$ are continuous.

$\therefore$ $X_+ := \max(X, 0)$ and $X_- := -\min(X, 0)$ are measurable.

$X = X_+ - X_-$

$|X| = X_+ + X_-$

$\therefore$ X is measurable iff $X_\pm$ are.

If $f:\mathbb{R}^d \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable, then

$$\boxed{X_1, \dots, X_d \text{ are } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \Rightarrow f(X_1, \dots, X_d) \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable}}$$

Often useful in dealing with some discontinuous functions of random variables.

Eg. The function $i_\alpha(x) = \begin{cases} 1/x, & x \neq 0 \\ \alpha, & x = 0 \end{cases}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. (for any $\alpha \in \mathbb{R}$)

Take $\alpha = 0$. Then

$$\left. i_0^{-1}(x, \infty) = \begin{cases} (0, 1/x), & \text{if } x \geq 0 \\ [0, \infty) \cup (-\infty, 1/x) & \text{if } x < 0. \end{cases} \right\} \in \mathcal{B}(\mathbb{R}) \ \forall x.$$

$\therefore$ i_0 is Borel-measurable.

In general $i_\alpha = i_0 \cdot \mathbb{1}_{\mathbb{R}\setminus\{0\}} + \alpha \mathbb{1}_{\{0\}}$. $\therefore$ measurable. ///

$\therefore$ If $f:\Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, so is $i_\alpha \circ f(x) = \begin{cases} 1/f(x), & f(x) \neq 0 \\ \alpha, & f(x) = 0. \end{cases}$

Measurability and "Limits"

Proposition [9.14] If $f_n : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ are measurable, then so are

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n .$$

Very false for $C(\mathbb{R})$

Pf. $(\sup_n f_n)^{-1}(-\infty, t] = \{x \in \mathbb{R} : \sup_n f_n \le t\}$

$= \{x \in \mathbb{R} : f_n(x) \le t \ \forall n\}$

$= \bigcap_n \{f_n \le t\} = \bigcap_n f_n^{-1}(-\infty, t] \in \mathcal{F}$

$(\inf_n f_n)^{-1}[t, \infty) = \ldots$

$$\limsup_{n\to\infty} f_n = \inf_n \sup_{j \ge n} f_j$$

✓

Simple Functions

A function $\varphi : \Omega \to \mathbb{R}$ is called simple if $\varphi(\Omega)$ is a finite set.

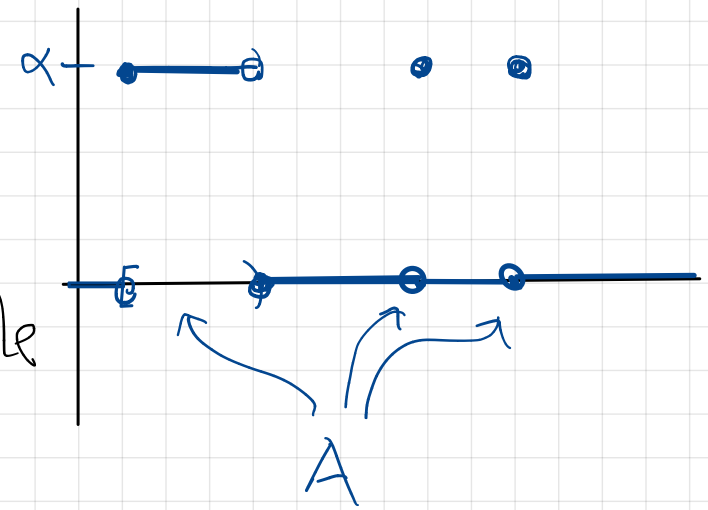
$$\varphi(\Omega) = \{\alpha_j : 1 \le j \le n\}$$

$A_j := \varphi^{-1}(\{\alpha_j\})$ are disjoint.

$$\varphi = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j}$$

E.g. $\varphi = \alpha \mathbb{1}_A$

$\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable iff $A \in \mathcal{F}$



Lemma: A simple function $\varphi : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable iff $\varphi^{-1}(\{\alpha\}) \in \mathcal{F} \quad \forall \alpha \in \mathbb{R}$.

$\varphi^{-1}(\{\alpha\}) = \phi$ for all but finitely many α.

$\Updownarrow$

$\sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j}$ measurable iff $A_1, \dots, A_n \in \mathcal{F}$.

disjoint

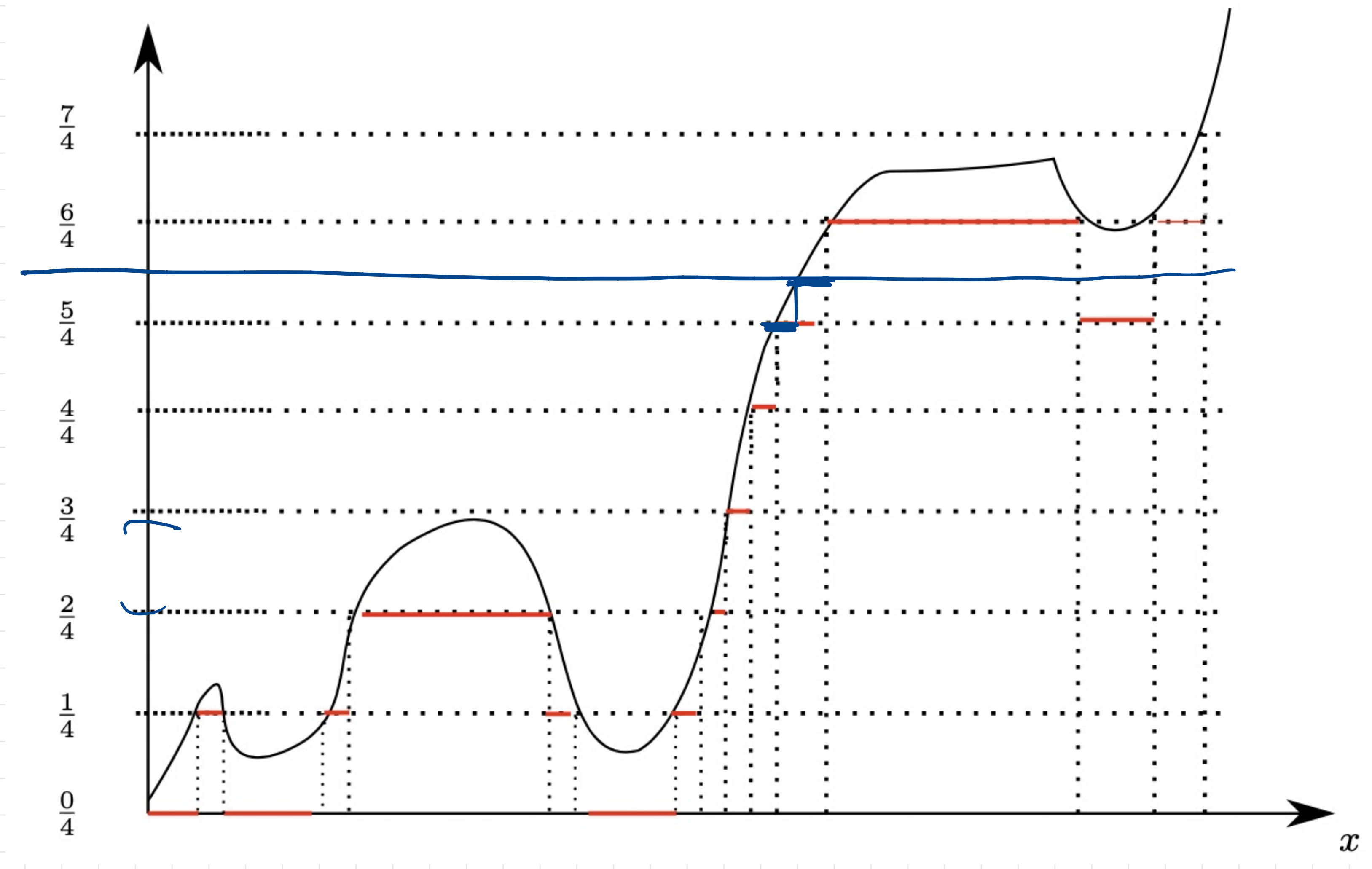
Simple Approximation

Theorem: [9.41] If $f:(\Omega,\mathcal{F})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is measurable, there is a sequence φ_n of simple measurable functions s.t.

$$\lim_{n\to\infty}\varphi_n(x)=f(x)\quad\forall x\in\Omega$$

Moreover, $\varphi_n\to f$ uniformly on $f^{-1}[-M,M]$ $\forall M>0$.

Pf. Suffices to assume $f\geq 0$ (then apply Thm to $f_\pm$)



$$\varphi_n(x)=\sum_{k=0}^{2^{2n}-1}\frac{k}{2^n}\mathbb{1}_{\{\frac{k}{2^n}<f\leq\frac{k+1}{2^n}\}}(x)+2^n\mathbb{1}_{\{f>2^n\}}(x)$$

$f^{-1}\left(\frac{k}{2^n},\frac{k+1}{2^n}\right]$

- $\varphi_n\leq\varphi_{n+1}\leq f$
- By construction, $0\leq f(x)-\varphi_n(x)\leq 2^{-n}$ on $\{f\leq 2^n\}$

$\therefore\ \varphi_n\to f$ unif. ///

Factoring Random Variables (Doob-Dynkin Representation)

$f : \mathbb{R}^d \to \mathbb{R}$ Borel measurable, and $X_1, \dots, X_d$ are $\mathcal{F} / \mathcal{B}(\mathbb{R})$ measurable $\Rightarrow$ $f(X_1, \dots, X_d)$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable.

Cor. [9.43] $X_1, \dots, X_d$ $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

& $\quad Y : \Omega \to \mathbb{R}$ $\quad \sigma(X_1, \dots, X_d)/\mathcal{B}(\mathbb{R})$-measurable

$\Rightarrow \exists$ Borel measurable $f : \mathbb{R}^d \to \mathbb{R}$ s.t. $Y = f(X_1, \dots, X_d)$.

To prove this, we need another characterization of $\sigma(X_1, \dots, X_d)$.

Lemma: Let $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_d \end{bmatrix} : \Omega \to \mathbb{R}^d$. Then

$$\sigma(X_1, \dots, X_d) = \underline{X}^*(\mathcal{B}(\mathbb{R}^d)) = \{ \underline{X}^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^d) \}$$

$\shortparallel$

smallest σ-field over Ω wrt which $X_1, \dots, X_d$ are measurable (with respect to $\mathcal{B}(\mathbb{R})$)

Pf. [HW].

Cor. [9.43] $X_1, \dots, X_d$ $\mathcal{F}/\mathcal{B}(\mathbb{R})$ - measurable

& $Y : \Omega \to \mathbb{R}$ $\sigma(X_1, \dots, X_d)/\mathcal{B}(\mathbb{R})$ - measurable

$\Rightarrow \exists$ Borel measurable $f : \mathbb{R}^d \to \mathbb{R}$ s.t. $Y = f(X_1, \dots, X_d)$.

Pf. To begin, assume $Y = \mathbb{1}_A$ $\therefore A \in \sigma(X_1, \dots, X_d) = \underline{X}^* \mathcal{B}(\mathbb{R}^d)$

$A = \underline{X}^{-1}(B)$ $\exists B \in \mathcal{B}(\mathbb{R}^d)$

$Y = \mathbb{1}_{\underline{X}^{-1}(B)} = \mathbb{1}_B \circ \underline{X}$ $\quad f = \mathbb{1}_B$ ✓

$$Y = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j} \qquad \therefore A_j \in \sigma(X_1, \dots, X_d) = \underline{X}^*(\mathcal{B}(\mathbb{R}^d))$$

$A_j = \underline{X}^{-1}(B_j)$, $B_j \in \mathcal{B}(\mathbb{R}^d)$

$$= \sum_{j=1}^{n} \alpha_j \mathbb{1}_{\underline{X}^{-1}(B_j)}$$

$$= \sum_{j=1}^{n} \alpha_j \mathbb{1}_{B_j} \circ \underline{X} = f(\underline{X}), \qquad f = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{B_j}$$

$\therefore$ Define $f := \limsup_{n\to\infty} f_n$

In general, $\exists$ Y_n of simple functions s.t. $Y_n \to Y$.

$\therefore Y = \lim_{n\to\infty} Y_n = \limsup_{n\to\infty} Y_n = \limsup_{n\to\infty} f_n(\underline{X}) = f(\underline{X})$ ///