Preimage Set Map

Let $f : \Omega \to S$ be a function between sets.

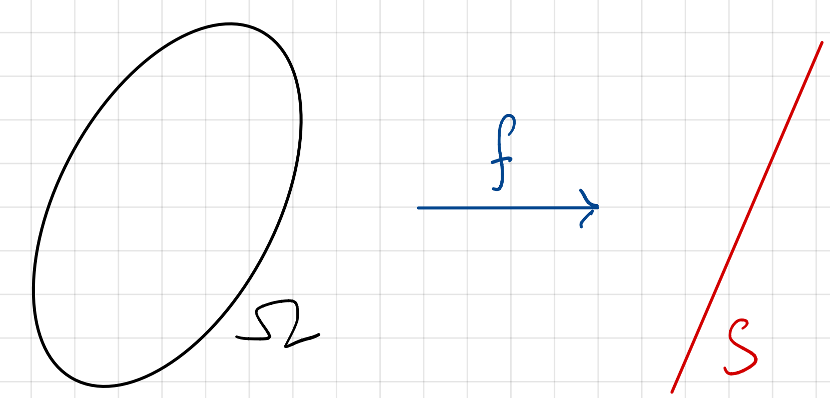
It induces a map $2^\Omega \to 2^S$ (image)

and a map $2^S \to 2^\Omega$ (preimage)

The image set map is not very well-behaved.
But preimage $B \mapsto f^{-1}(B)$ is very well-behaved. [Driver, §4.2]

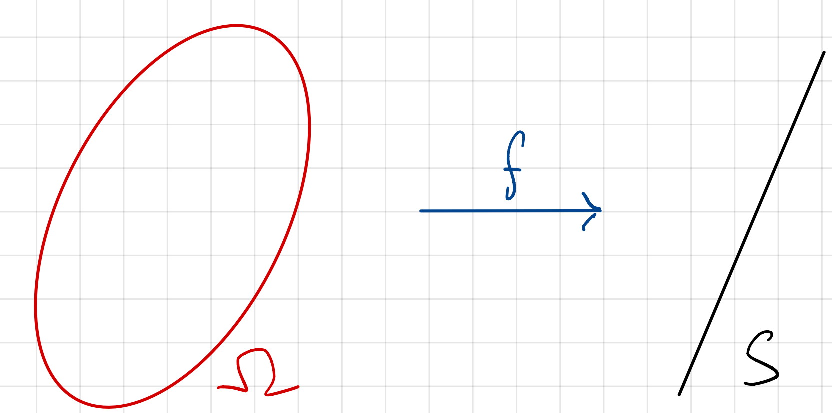
Def: Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{B})$ be measurable spaces.
The pull-back of $\mathcal{B}$ to Ω is

$$f^*\mathcal{B} := \{ f^{-1}(B) \subseteq \Omega : B \in \mathcal{B} \}$$



The push-forward of $\mathcal{F}$ to S is

$$f_*\mathcal{F} := \{ E \subseteq S : f^{-1}(E) \in \mathcal{F} \}$$



Both of them are σ-fields [HW].

$f : \Omega \to S$.

More generally, $f^*\mathcal{E} = \{ f^{-1}(E) \subseteq \Omega : E \in \mathcal{E} \} \subseteq 2^\Omega$ makes sense for any subset $\mathcal{E} \subseteq 2^S$.

Lemma: $\sigma(f^*\mathcal{E}) = f^*(\sigma(\mathcal{E}))$.

Pf.

Def: $(\Omega, \mathcal{F})$, $(S, \mathcal{B})$ measurable spaces)

$f : \Omega \longrightarrow S$ is $(\mathcal{F} / \mathcal{B})$ - measurable if $f^* \mathcal{B} \subseteq \mathcal{F}$.

I.e. $\forall B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{F}$.

E.g. Indicator functions $\mathbb{1}_A : \Omega \to \mathbb{R}$, $A \subseteq \Omega$.

Prop: Let $\mathcal{E} \subseteq \mathcal{B}$ s.t. $\sigma(\mathcal{E}) = \mathcal{B}$. Then f is measurable iff $f^* \mathcal{E} \subseteq \mathcal{F}$.

Pf. $(\Longrightarrow)$ Follows b/c $\mathcal{E} \subseteq \mathcal{B}$.

$(\Longleftarrow)$

Important Example:

$X: \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable iff

Def: Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a (Borel) random variable is a $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable function $X: \Omega \to \mathbb{R}$.

Prop: Compositions of measurable functions are measurable.

Cor: Let $X_1, X_2, \dots, X_d$ be random variables on $(\Omega, \mathcal{F})$.
If $f: \mathbb{R}^d \to \mathbb{R}$ is continuous (or more generally Borel measurable)
then $Y = f(X_1, \dots, X_d)$ is a random variable. (E.g. $X_1 + X_2$, $X_1 X_2$, ...)

Pf.

$(\Omega, \mathcal{F}, \mathbb{P})$ "background" probability space
(inaccessible).

$X_1, X_2, \dots, X_d$ r.v.'s on Ω. What is actually knowable?

Two things.

1. "Current information".

<u>Def</u>: The σ-field generated by $\{X_j\}_{j=1}^{d}$ is

$$\sigma(X_1, \dots, X_d) := \sigma\Big(\bigcup_j X_j^* \mathcal{B}(\mathbb{R})\Big)$$

2. "Distribution"

Prop: If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $f: \Omega \to S$ is measurable (wrt $(S, \mathcal{B})$),

then $\mu_f : \mathcal{B} \to [0,1]$

$$\mu_f = \mathbb{P} \circ f^{-1}$$

is a probability measure on $(S, \mathcal{B})$.

Pf.

Special case: $(S, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Here $f = X$ is a r.v.

$\mu_X = \mathbb{P} \circ X^{-1}$ i.e. $\mu_X(B) = \mathbb{P}(X \in B)$, $B \in \mathcal{B}(\mathbb{R})$

So