# Preimage Set Map

Let $f:\Omega\to S$ be a function between sets.

It induces a map $2^\Omega \to 2^S$ (image) $\Omega \supseteq A \longmapsto f(A) = \{f(\omega) : \omega\in A\} \subseteq S$

and a map $2^S \to 2^\Omega$ (preimage) $S \supseteq E \longmapsto f^{-1}(E) = \{\omega\in\Omega : f(\omega)\in E\} \subseteq \Omega$

The image set map is not very well-behaved.

But preimage $B \mapsto f^{-1}(B)$ is very well-behaved.

$f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$

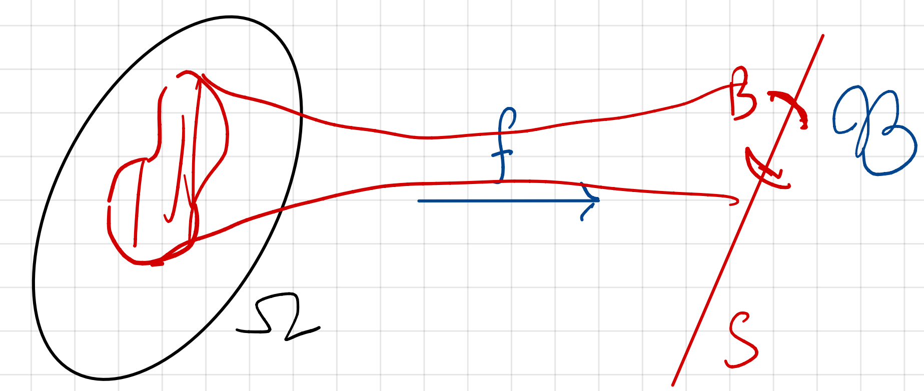
also $f^{-1}(E^c) = f^{-1}(E)^c$

[Driver, §4.2]

Def: Let $(\Omega,\mathcal{F})$ and $(S,\mathcal{B})$ be measurable spaces.

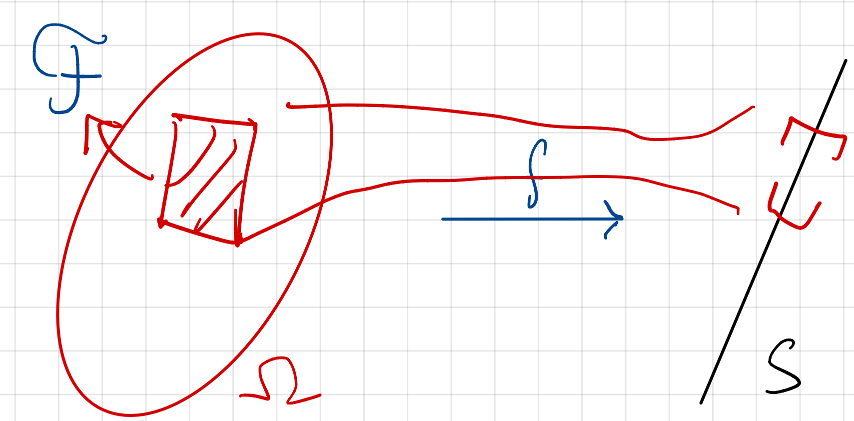
The pull-back of $\mathcal{B}$ to $\Omega$ is

$$f^*\mathcal{B} := \{f^{-1}(B)\subseteq\Omega : B\in\mathcal{B}\}$$




The push-forward of $\mathcal{F}$ to $S$ is

$$f_*\mathcal{F} := \{E\subseteq S : f^{-1}(E)\in\mathcal{F}\}$$




Both of them are $\sigma$-fields [HW].

$f: \Omega \to S$.

More generally, $f^*\mathcal{E} = \{ f^{-1}(E) \subseteq \Omega : E \in \mathcal{E} \} \subseteq 2^{\Omega}$ makes sense for any subset $\mathcal{E} \subseteq 2^S$.

**Lemma:** $\sigma(f^*\mathcal{E}) = f^*(\sigma(\mathcal{E}))$.

**Pf.** By HW, $f^*(\sigma(\mathcal{E}))$ is a $\sigma$-field. Since $\mathcal{E} \subseteq \sigma(\mathcal{E})$

$\therefore f^*\mathcal{E} \subseteq f^*(\sigma(\mathcal{E}))$

$\therefore \sigma(f^*\mathcal{E}) \subseteq$

For $\supseteq$: $\mathcal{G} = \sigma(f^*\mathcal{E})$.

Then $f_*\mathcal{G}$ is a $\sigma$-field.

If $E \in \mathcal{E}$, $f^{-1}(E) \in f^*\mathcal{E} \subseteq \sigma(f^*\mathcal{E}) = \mathcal{G}$. $\therefore E \in f_*\mathcal{G}$.

$\therefore \mathcal{E} \subseteq f_*\mathcal{G}$.

$\therefore \sigma(\mathcal{E}) \subseteq$

$\therefore B \in \sigma(\mathcal{E})$

$\therefore B \in f_*\mathcal{G}$ i.e. $f^{-1}(B) \in \mathcal{G}$, i.e. $f^*(\sigma(\mathcal{E})) \subseteq \mathcal{G} = \sigma(f^*\mathcal{E})$

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**Def**: $(\Omega, \mathcal{F})$, $(S, \mathcal{B})$ measurable spaces)

$f : \Omega \longrightarrow S$ is $(\mathcal{F}/\mathcal{B})$-measurable if $f^*\mathcal{B} \subseteq \mathcal{F}$.

I.e. $\forall B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{F}$.

E.g. Indicator functions $\mathbb{1}_A : \Omega \to \mathbb{R}$, $A \subseteq \Omega$.

$\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable iff $A \in \mathcal{F}$.

$$\mathbb{1}_A^{-1}(B) = \begin{cases} \emptyset & \text{if } 0,1 \notin B \\ \Omega & \text{if } 0,1 \in B \\ A & \text{if } 1 \in B,\ 0 \notin B \\ A^c & \text{if } 1 \notin B,\ 0 \in B \end{cases}$$

**Prop**: Let $\mathcal{E} \subseteq \mathcal{B}$ s.t. $\sigma(\mathcal{E}) = \mathcal{B}$. Then $f$ is measurable iff $f^*\mathcal{E} \subseteq \mathcal{F}$.

**Pf.** $(\Rightarrow)$ Follows b/c $\mathcal{E} \subseteq \mathcal{B}$.

$(\Leftarrow)$ $f^*\mathcal{B} = f^*(\sigma(\mathcal{E})) = \sigma(f^*\mathcal{E})$. $\sigma$-field.

$f^*\mathcal{E} \subseteq \mathcal{F}$

$\therefore \sigma(f^*\mathcal{E}) \subseteq \mathcal{F}$. ///

## Important Example:

$X:\Omega\to\mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable iff

- $X^{-1}(B)\in\mathcal{F}\quad\forall B\in\mathcal{B}(\mathbb{R})$

or

- $X^{-1}(-\infty,t]\in\mathcal{F}\quad\forall t\in\mathbb{R}$ $\quad\longleftarrow\quad$ $X^{-1}(-\infty,t]=\{\omega\in\Omega: X(\omega)\le t\}$ $=\{X\le t\}$
- $X^{-1}(-\infty,t)\in\mathcal{F}$
- $X^{-1}(a,b]\in\mathcal{F}\quad\forall\ a<b\in\mathbb{R}$ $\quad\longleftrightarrow\quad$ $\{X\in(a,b]\}=\{a<X\le b\}$
- $X^{-1}(a,\infty)\in\mathcal{F}\quad\forall a\in\mathbb{R}$
- $X^{-1}[a,\infty)\in\mathcal{F}$

$\vdots$

**Def:** Given a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a (Borel) random variable is a $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable function $X:\Omega\to\mathbb{R}$.

**Prop:** Compositions of measurable functions are measurable.

$$\begin{tikzcd}
(\Omega,\mathcal{F}) \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & (S,\mathcal{B}) \arrow[r, "g"] & (T,\mathcal{C})
\end{tikzcd}$$

$$\forall C\in\mathcal{C} \quad (g\circ f)^{-1}(C) = f^{-1}\big(\underbrace{g^{-1}(\underset{\in\mathcal{C}}{C})}_{\therefore\in\mathcal{B}}\big) \therefore \in \mathcal{F}. \qquad ///$$

**Cor:** Let $X_1, X_2, \dots, X_d$ be random variables on $(\Omega,\mathcal{F})$.
If $f:\mathbb{R}^d\to\mathbb{R}$ is continuous (or more generally Borel measurable)
then $Y=f(X_1,\dots,X_d)$ is a random variable. (E.g. $X_1+X_2$, $X_1X_2$, ...)

**Pf.** If $f$ is continuous, $f^{-1}(-\infty,t)$ is open in $\mathbb{R}^d$ $\therefore$ also in $\mathcal{B}(\mathbb{R}^d)$
$\therefore$ $f$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$-measurable.

Define $\underline{X} = (X_1,\dots,X_d):\Omega\to\mathbb{R}^d$.

$$\underline{X}^{-1}\big((-\infty,t_1]\times(-\infty,t_2]\times\cdots\times(-\infty,t_d]\big) = \{\omega\in\Omega : X_j(\omega)\in(-\infty,t_j],\ 1\le j\le d\}$$

$$= \underbrace{\{X_1\le t_1\}}_{\in\mathcal{F}}\cap\underbrace{\{X_2\le t_2\}}_{\in\mathcal{F}}\cap\cdots\cap\underbrace{\{X_d\le t_d\}}_{\in\mathcal{F}}$$

$\therefore \in \mathcal{F}$.

$(\Omega, \mathcal{F}, \mathbb{P})$ "background" probability space (inaccessible).

$X_1, X_2, \dots, X_d$ r.v.'s on $\Omega$. What is actually knowable?

Two things.

1. "Current information".

**Def:** The $\sigma$-field generated by $\{X_j\}_{j=1}^d$ is

$$\sigma(X_1, \dots, X_d) := \sigma\Big(\bigcup_j X_j^* \mathcal{B}(\mathbb{R})\Big) \subseteq \mathcal{F}$$

the smallest $\sigma$-field wrt which $X_1, \dots, X_d$ are measurable.

## 2. "Distribution"

**Prop:** If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $f : \Omega \to S$ is measurable (wrt $(S, \mathcal{B})$), then

$$\mu_f : \mathcal{B} \to [0,1]$$

$$\mu_f = \mathbb{P} \circ f^{-1} = f^* \mathbb{P} \qquad \mu_f(B) = \mathbb{P} \circ f^{-1}(B) = \mathbb{P}(f^{-1}(B)) = \mathbb{P}\{f \in B\}$$

is a probability measure on $(S, \mathcal{B})$.

**Pf.**

$$\mu_f(S) = \mathbb{P}(f^{-1}(S)) = \mathbb{P}(\Omega) = 1.$$

$$\mu_f\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \mathbb{P}\left(f^{-1}\left(\bigsqcup_{n=1}^{\infty} B_n\right)\right) = \mathbb{P}\left(\bigsqcup_{n=1}^{\infty} f^{-1}(B_n)\right) = \sum_{n=1}^{\infty} \mathbb{P}(f^{-1}(B_n))$$

$$= \sum_{n=1}^{\infty} \mu_f(B_n). \quad ///$$

**Special case:** $(S, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Here $f = X$ is a r.v.

$$\mu_X = \mathbb{P} \circ X^{-1} \quad \text{i.e.} \quad \mu_X(B) = \mathbb{P}(X \in B)$$

$$B \in \mathcal{B}(\mathbb{R})$$

So $\mu_X(-\infty, t] = \mathbb{P}(X \le t) = F_X(t)$.