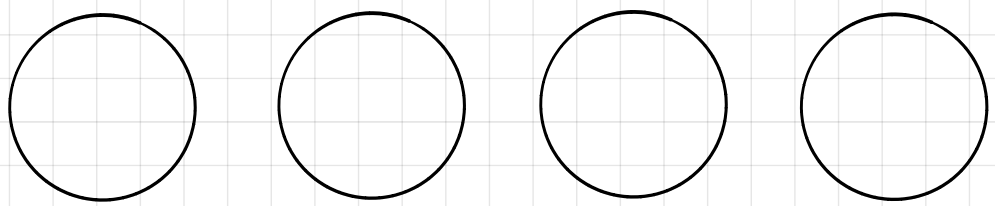
Random Variables: Motivation

"Experiments"

Eg. Toss a fair coin N times.



Eg. Throw a dart at a board of radius R.

- Each experiment has an **outcome**.
- The set of all possible outcomes is the **sample space** Ω.
- "Probability" is a measure of the likelihood of a **set of outcomes** = an **event** $E \subseteq \Omega$.

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space.

Outcomes are elements $\omega \in \Omega$.
Events are subsets of Ω, in $\mathcal{F}$.

A random variable is a function $X : \Omega \to S$
(Probably should call them "random functions"
but the very old "variable" terminology has
stuck since used by Laplace in the early 19^{th}
Century.)

"state space"
usually $\mathbb{R}$; could be $\mathbb{C}$
(could be $\mathbb{R}^d$; then usually
call X a "random vector")

Need to be able to calculate probabilities of events like

$$\{X \leq 1\}$$

Def: A function $X : \Omega \to \mathbb{R}$ is a random variable
if $\{X \leq t\} \in \mathcal{F}$ for all $t \in \mathbb{R}$.

CDFs (Again)

$X : \Omega \to \mathbb{R}$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

Define $F_X : \mathbb{R} \to \mathbb{R} : \quad F_X(t) = \mathbb{P}(X \leq t)$

Proposition: F_X is non-decreasing, right-continuous, and

$$\lim_{t \to -\infty} F_X(t) = 0, \quad \lim_{t \to +\infty} F_X(t) = 1.$$

Pf.

$\therefore$ F_X is the CDF of a unique Borel probability measure μ_X on $\mathbb{R}$

$\uparrow$

The probability distribution of X.

Often μ_X is all we'll really know about X.

And more often, we won't even know μ_X, but will only have some limited clues about it.

(Great) Expectations

Eg. Finite sample space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$

(May as well have $\mathcal{F} = 2^\Omega$.)

Then $\quad \mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\{\omega\}) =$

If $X : \Omega \to \mathbb{R}$ is a random variable,

$$F_X(t) = \mathbb{P}(X \le t) =$$

Can we get a "snapshot" number that tells us <u>something</u> about this distribution?

↳ weighted average: $\quad \mathbb{E}(X) :=$

Eg. Toss a fair coin 3 times; $X = \#$ Heads.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$$E(X) = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\{\omega\}) \qquad (\star)$$

Makes perfect sense if Ω is finite. Also okay if Ω is countable, But won't help us if $\mathbb{P}(\{\omega\}) = 0$ for all $\omega \in \Omega$.

Undergraduate Probability Approach:

$$\sum_{\omega} X(\omega)\, \mathbb{P}(\{\omega\}) =$$

Problems: Many.

We will develop the right generalization of $(\star)$ to work in any probability space: the Lebesgue Integral

$$E(X) = \int_{\Omega} X \, d\mathbb{P}.$$