# Lebesgue Measure (§ 6.6 in Driver)

The Radon measure on $\mathbb{R}$ satisfying

$$\lambda((a,b]) = b-a, \qquad -\infty < a < b < \infty$$

is called **Lebesgue measure**. It is the most important measure on $\mathbb{R}$.

Notice: if $\tau \in \mathbb{R}$, $J = (a,b] \in \mathcal{A}_{(\,]}$, then

$$\lambda(J+\tau) =$$

$$\Rightarrow \lambda(A+\tau) = \qquad \text{for } A \in \mathcal{B}_{(\,]}(\mathbb{R})$$

$$\Rightarrow \lambda^*(E+\tau) = \qquad \text{for \underline{all} } E \subseteq \mathbb{R}$$ [HW3]

Theorem: $\lambda$ is the unique translation invariant Borel measure s.t. $\lambda((0,1]) = 1$; if $\mu$ is another translation invariant Borel measure, then $\mu =$

Pf. Since $\lambda^*$ is translation invariant, $\lambda = \lambda^*|_{\mathcal{B}(\mathbb{R})}$ is too, and $\lambda((0,1]) = 1 - 0 = 1$.

For the converse:

By similar reasoning: If $\gamma \in \mathbb{R}$, $B \in \mathcal{B}(\mathbb{R})$,

$$\lambda(\gamma \cdot B) =$$

Pf.

## Null Sets (§6.10 in Driver)

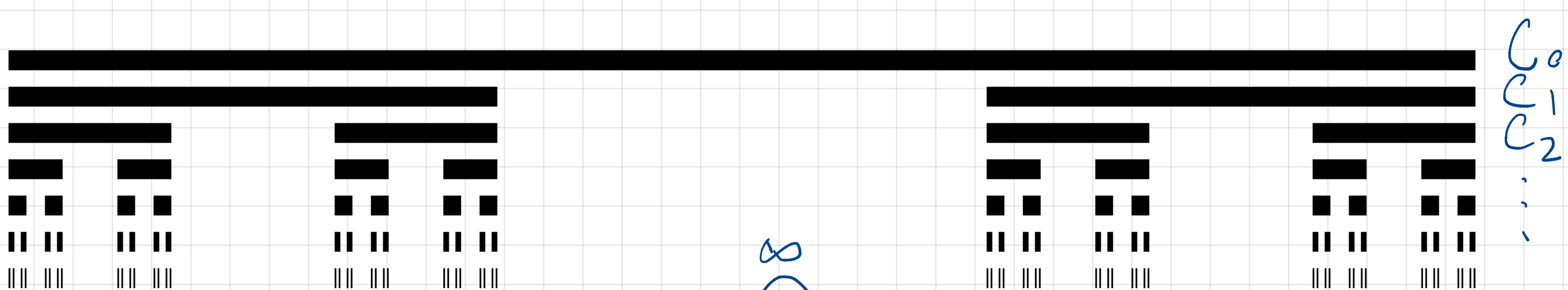
In a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable set $N \in \mathcal{F}$ is a **null set** if $\mu(N) = 0$.

Eg. If $\mu = \delta_{x_0}$, any set not containing $x_0$ is null.

Lebesgue null sets:

- If $X \subset \mathbb{R}$ is countable, then $\lambda(X) = 0$.

- There are lots of uncountable nullsets, too!




$$C = \bigcap_{n=0}^{\infty} C_n$$

Problem: most subsets of the Cantor set are not Borel sets.

That is: there are many sets $N \in \mathcal{B}(\mathbb{R})$ of Lebesgue measure $0$ that contain non-Borel sets $\Lambda \subseteq N$. This can sometimes cause technical problems.

Definition: A measure space $(\Omega, \mathcal{F}, \mu)$ is called (null) complete if, for every $N \in \mathcal{F}$ with $\mu(N) = 0$, every subset $\Lambda \subseteq N$ is in $\mathcal{F}$ (and $\therefore$ $\mu(\Lambda) = 0$).

Theorem: For any measure space $(\Omega, \mathcal{F}, \mu)$, there is an extension $\tilde{\mathcal{F}} \supseteq \mathcal{F}$, $\tilde{\mu}|_{\mathcal{F}} = \mu$, s.t. $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is

Pf. The most obvious thing actually works:

$$\tilde{\mathcal{F}} = \{A \cup \Lambda : A \in \mathcal{F},\ \Lambda \subset \Omega,\ \exists N \in \mathcal{F} \text{ s.t. } \mu(N) = 0 \ \& \ \Lambda \subseteq N\}$$

$$\tilde{\mu}(A \cup \Lambda) := \mu(A).$$

- $\tilde{\mathcal{F}}$ is a $\sigma$-field containing $\mathcal{F}$:

$\tilde{\mathcal{F}} = \{ A \cup \Lambda : A \in \mathcal{F}, \Lambda \subseteq \Omega, \exists N \in \mathcal{F} \text{ s.t. } \mu(N) = 0 \ \& \ \Lambda \subseteq N \}$

$\tilde{\mu}(A \cup \Lambda) := \mu(A)$.

- $\tilde{\mu}$ is well-defined:

- $\tilde{\mu}$ is a measure.

- $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is (null) complete (by construction).

# The Lebesgue $\sigma$-Field.

Often, when one speaks of Lebesgue measure, one implies a particular large $\sigma$-field:

$$\mathcal{M} := \{E \subseteq \mathbb{R} : \forall A \subseteq \mathbb{R} \quad \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\}$$

↳ $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{M}$

↳ $\mathcal{M}$ is null complete (and bigger than $\tilde{\mathcal{B}}(\mathbb{R})$)

We will never use this $\mathcal{M}$. For us, Lebesgue measure is a Borel measure; worst case, we may need to complete the Borel $\sigma$-field in some applications.

BTW, fun fact:

In a finite (pre)measure space $(\Omega, \mathcal{A}, \mu)$,

$$\mathcal{M}_\mu = \bar{\mathcal{A}} \quad \text{(the closure in the pseudo-metric } d_\mu\text{)}$$

[Driver, Prop. 7.11]