Lebesgue Measure (§ 6.6 in Driver)

The Radon measure on $\mathbb{R}$ satisfying

$$\lambda((a,b]) = b-a, \qquad -\infty < a < b < \infty$$

is called Lebesgue measure. It is the most important measure on $\mathbb{R}$.

Notice: if $\tau \in \mathbb{R}$, $J = (a,b] \in \mathcal{J}_{(\,]}$, then

$$\lambda(J+\tau) = \lambda((a+\tau, b+\tau]) = (b+\cancel{\tau}) - (a+\cancel{\tau}) = b-a = \lambda(J).$$

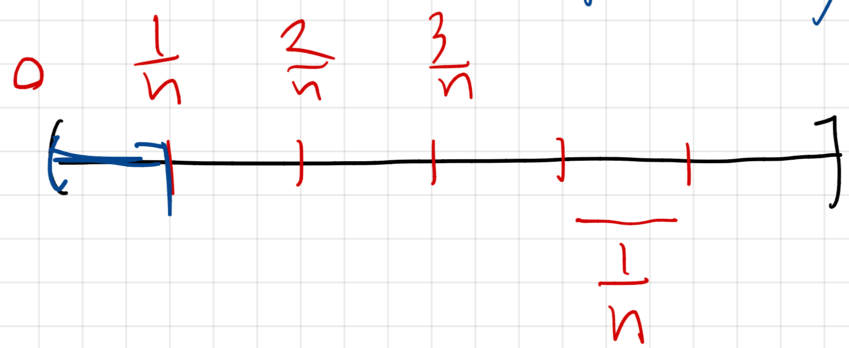
$\Rightarrow \lambda(A+\tau) = \lambda(A)$ for $A \in \mathcal{B}_{(\,]}(\mathbb{R})$

$\Rightarrow \lambda^*(E+\tau) = \lambda^*(E)$ for all $E \subseteq \mathbb{R}$ [HW3]

Theorem: λ is the unique translation invariant Borel measure s.t. $\lambda((0,1]) = 1$; if μ is another translation invariant Borel measure, then $\mu = \alpha\lambda$ for some $\alpha \geq 0$.

Pf. Since λ^* is translation invariant, $\lambda = \lambda^*|_{\mathcal{B}(\mathbb{R})}$ is too,
and $\lambda((0,1]) = 1-0 = 1$.

For the converse: S'pose $\mu((0,1]) = 1$, μ is trans. inv.,



$$(0,1] = \bigsqcup_{k=0}^{n-1} \left(\frac{k}{n}, \frac{k+1}{n}\right] = \bigsqcup_{k=0}^{n-1} \left(\frac{k}{n} + \left(0, \frac{1}{n}\right]\right)$$

$$\therefore \overset{1}{\overset{\|}{\mu(0,1]}} = \sum_{k=0}^{n-1} \underbrace{\mu\left(\frac{k}{n} + \left(0,\frac{1}{n}\right]\right)}_{\mu(0,\frac{1}{n}]} = n\cdot \mu\left(0,\frac{1}{n}\right]$$

$\therefore \mu(0,\frac{1}{n}] = \frac{1}{n}$.

$\Downarrow$

$\mu(0,\frac{m}{n}] = \frac{m}{n}$. $m,n \in \mathbb{N}$

$F_\mu(\frac{m}{n}) - F_\mu(0) = \frac{m}{n}$

$\forall r \in \mathbb{Q}$ $F_\mu(r) - F_\mu(0) = r$. $\Rightarrow$ $\therefore F_\mu(x) - F_\mu(0) = x$ $\therefore \mu = \lambda$.

if $\mu(0,1] = \alpha$ $\begin{cases} = 0, & \mu(\mathbb{R}) = \mu\left(\bigsqcup_{n=-\infty}^{\infty}((0,1]+n)\right) = \sum_{n=-\infty}^{\infty} 0 = 0. \\ > 0, & \therefore \frac{1}{\alpha}\mu \text{ is trans. inv. and } \left(\frac{1}{\alpha}\mu\right)((0,1]) = \frac{1}{\alpha}\cdot\alpha = 1 \quad \therefore \frac{1}{\alpha}\mu = \lambda. \end{cases}$

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By similar reasoning: If $\gamma \in \mathbb{R}$, $B \in \mathcal{B}(\mathbb{R})$,

$$\lambda(\gamma \cdot B) = |\gamma| \cdot \lambda(B)$$

$\gamma \cdot B = \{\gamma \cdot x : x \in B\}$

Pf. If $\gamma = 0$, $\lambda(\{0\}) = 0 = |0| \cdot \lambda(B)$

If $\gamma \neq 0$, $\mu(B) := \frac{1}{|\gamma|} \lambda(\gamma \cdot B)$ $\leftarrow$ This is a transl. inv. measure. (HW)

$\therefore$ $\mu(0,1] = \frac{1}{|\gamma|} \lambda(\gamma \cdot (0,1])$

$\hookrightarrow$ $\gamma > 0$, $\lambda(0,\gamma] = \gamma \rightsquigarrow \mu(0,1] = 1$

$\hookrightarrow$ $\gamma < 0$, $\lambda([\gamma, 0)) = \lambda(\{\gamma\}) + \lambda(\gamma, 0)$

$$= \cancel{\lambda(\{\gamma\})} + \lambda(\gamma, 0] - \cancel{\lambda\{0\}}$$

$$= 0 - \gamma = |\gamma|$$

$\therefore$ $\mu = \lambda$.

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Null Sets (§6.10 in Driver)

In a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable set $N \in \mathcal{F}$ is a **null set** if $\mu(N) = 0$.

Eg. If $\mu = \delta_{x_0}$, any set not containing x_0 is null.

Lebesgue null sets:

- If $X \subset \mathbb{R}$ is countable, then $\lambda(X) = 0$.

$$X = \{x_1, x_2, x_3, \dots\}$$

$$X \subseteq \bigcup_{n=1}^{\infty} \left(x_n - \frac{\varepsilon}{2^n}, x_n\right] \quad \text{for any } \varepsilon > 0.$$

$$\lambda(X) \le \sum_{n=1}^{\infty} \lambda\left(x_n - \frac{\varepsilon}{2^n}, x_n\right] = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

$\therefore \lambda(X) = 0.$

- There are lots of uncountable nullsets, too!

C_0
C_1
C_2
$\vdots$

$\lambda(C_0) = 1$
$\lambda(C_1) = 2/3$
$\lambda(C_2) = 4/9$

C_n is a $\bigsqcup$ of 2^n intervals of length $\frac{1}{3^n}$.

$\therefore \lambda(C_n) = \left(\frac{2}{3}\right)^n$

$$C = \bigcap_{n=0}^{\infty} C_n \quad \therefore \lambda(C) \le \left(\frac{2}{3}\right)^n \ \forall n \ge 0. \Rightarrow \lambda(C) = 0.$$

Problem: most subsets of the Cantor set are not Borel sets.

That is: there are many sets $N \in \mathcal{B}(\mathbb{R})$ of Lebesgue measure 0 that contain non-Borel sets $\Lambda \subseteq N$. This can sometimes cause technical problems.

Definition: A measure space $(\Omega, \mathcal{F}, \mu)$ is called (null) complete if, for every $N \in \mathcal{F}$ with $\mu(N) = 0$, every subset $\Lambda \subseteq N$ is in $\mathcal{F}$ (and $\therefore$ $\mu(\Lambda) = 0$).

Theorem: For any measure space $(\Omega, \mathcal{F}, \mu)$, there is an extension $\tilde{\mathcal{F}} \supseteq \mathcal{F}$, $\tilde{\mu}|_{\mathcal{F}} = \mu$, s.t. $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is

Pf. The most obvious thing actually works:

$$\tilde{\mathcal{F}} = \{ A \cup \Lambda : A \in \mathcal{F},\ \Lambda \subseteq \Omega,\ \exists N \in \mathcal{F} \text{ s.t. } \mu(N) = 0 \ \&\ \Lambda \subseteq N \}$$

$$\tilde{\mu}(A \cup \Lambda) := \mu(A).$$

- $\tilde{\mathcal{F}}$ is a σ-field containing $\mathcal{F}$: $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ $\therefore$ $\emptyset, \Omega \in \tilde{\mathcal{F}}$.

- $$(A \cup \underset{\substack{\cap \\ N}}{\Lambda})^c = A^c \cap \underset{\substack{\| \\ (N \setminus \Lambda) \cup N^c}}{\Lambda^c} = \underbrace{(A^c \cap (N \setminus \Lambda))}_{\substack{\cap \\ N}} \cup (\underbrace{A^c \cap N^c}_{\in \mathcal{F}}) \in \tilde{\mathcal{F}}.$$

- ~~If~~ $$\underbrace{\bigcup_n (\underset{\substack{\cap \\ \mathcal{F}}}{A_n} \cup \underset{\substack{\cap \\ N_n}}{\Lambda_n})}_{\therefore \in \tilde{\mathcal{F}}} = \underset{\in \mathcal{F}}{\bigcup_n A_n} \cup \underset{\substack{\cap \\ \bigcup_n N_n \in \mathcal{F}}}{\bigcup_n \Lambda_n} \qquad \mu\Big(\bigcup_n N_n\Big) \le \sum_{n=1}^{\infty} \underset{0}{\mu(N_n)} = 0.$$

$$\tilde{\mathcal{F}} = \{A \cup \Lambda : A \in \mathcal{F},\ \Lambda \subseteq \Omega,\ \exists N \in \mathcal{F} \text{ s.t. } \mu(N) = 0 \ \&\ \Lambda \subseteq N\}$$

$$\tilde{\mu}(A \cup \Lambda) := \mu(A).$$

- $\tilde{\mu}$ is well-defined: Suppose $A, A', N, N' \in \mathcal{F}$, $\mu(N) = \mu(N') = 0$, $N \supseteq \Lambda$, $N' \supseteq \Lambda'$

$$A \cup \Lambda = A' \cup \Lambda'.$$

$$A \subseteq A \cup \Lambda \subseteq A \cup \Lambda \cup N' = A' \cup \underline{\Lambda' \cup N'} = A' \cup N'.$$

$$\therefore \mu(A) \le \mu(A' \cup N') \le \mu(A') + \underbrace{\cancel{\mu(N')}}_{0} = \mu(A')$$

$$\longleftrightarrow \quad \mu(A') \le \mu(A)$$

- $\tilde{\mu}$ is a measure. $A_n \cup \Lambda_n$ disjoint, $A_n \in \mathcal{F}$, $\Lambda_n \subseteq N_n$, $\mu(N_n) = 0$.

$$\therefore \tilde{\mu}\Big(\bigsqcup_n A_n \cup \Lambda_n\Big) = \tilde{\mu}\Big(\bigsqcup_n A_n \cup \underbrace{\bigsqcup_n \Lambda_n}_{\text{is a subset of } \cup N_n \text{ is a null set}}\Big) = \mu\Big(\bigsqcup_n A_n\Big) = \sum_n \mu(A_n) = \sum_n \tilde{\mu}(A_n \cup \Lambda_n)$$

- $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is (null) complete (by construction).

The Lebesgue σ-Field.

Often, when one speaks of Lebesgue measure, one implies a particular large σ-field:

$$\mathcal{M} := \{E \subseteq \mathbb{R} : \forall A \subseteq \mathbb{R} \;\; \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\}$$

↳ $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{M}$

↳ $\mathcal{M}$ is null complete (and bigger than $\widetilde{\mathcal{B}}(\mathbb{R})$)

We will never use this $\mathcal{M}$. For us, Lebesgue measure is a Borel measure; worst case, we may need to complete the Borel σ-field in some applications.

BTW, fun fact:

In a finite (pre)measure space $(\Omega, \mathcal{A}, \mu)$,

$$\mathcal{M}_\mu = \bar{\mathcal{A}} \quad \text{(the closure in the pseudo-metric } d_\mu\text{)}$$

[Driver, Prop. 7.11]