If Ω is a topological space, any measure on $\mathcal{B}(\Omega)$ will be referred to as a Borel Measure.

We will principally work with Borel measures throughout this course.

Radon Measures (§ 6.5 in Driver)

A Borel measure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ on $\mathbb{R}$ is called a Radon measure if

$$\mu([a,b]) < \infty \quad \forall\, a<b \in \mathbb{R}.$$

(More generally, a Radon measure on a topological space Ω is a Borel measure μ s.t. $\mu(K) < \infty$ for all compact K (and satisfies some regularity conditions that turn out be be automatic when $\Omega = \mathbb{R}$).

Eg. The Stieltjes premeasures $\mu_F((a,b]) = F(b) - F(a)$ for $F: \mathbb{R} \to \mathbb{R}$ increasing and right continuous.

Theorem: If μ is a Radon measure on $\mathbb{R}$, then there exists a non-decreasing, right-continuous function $F:\mathbb{R}\to\mathbb{R}$ (unique up to an additive constant) s.t.

$$\mu((a,b]) = F(b)-F(a), \quad -\infty \le a \le b \le \infty.$$

Pf.

Right continuity:

We saw (Lecture 3.1) that $(\Omega, \mathcal{B}_{(\,]}, \mu_F)$ is a premeasure for every right-continuous increasing $F: \mathbb{R} \to \mathbb{R}$. $\therefore$ By the extension theorem, we now have a characterization of Radon measures on $\mathbb{R}$.

Re: Convergence of $\mu(A_n)$, more generally: [Driver, Prop. 6.3]

Prop: Let μ be a finitely additive measure on $(\Omega, \mathcal{A})$. TFAE:

(1) μ is a premeasure on $\mathcal{A}$.

(2) If $A_n, A \in \mathcal{A}$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.

Moreover, in the case $\mu(\Omega) < \infty$, the following are also equivalent:

(3) If $A_n \downarrow A$ in $\mathcal{A}$, then $\mu(A_n) \downarrow \mu(A)$.

(4) If $A_n \uparrow \Omega$ in $\mathcal{A}$, then $\mu(A_n) \uparrow \mu(\Omega)$.

(5) If $A_n \downarrow \emptyset$ in $\mathcal{A}$, then $\mu(A_n) \downarrow 0$.

Pf: (1) $\Rightarrow$ (2)

(2) $\Rightarrow$ (1)

To include (3-5), use the fact (true in the finite measure case) that

$$A \subseteq B \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$$

Def: Let μ be a Borel probability measure on $\mathbb{R}$.

$$F_\mu : \mathbb{R} \to \mathbb{R} \; ; \; F_\mu(x) = \mu((-\infty, x])$$

is the cumulative distribution function (CDF) of μ.

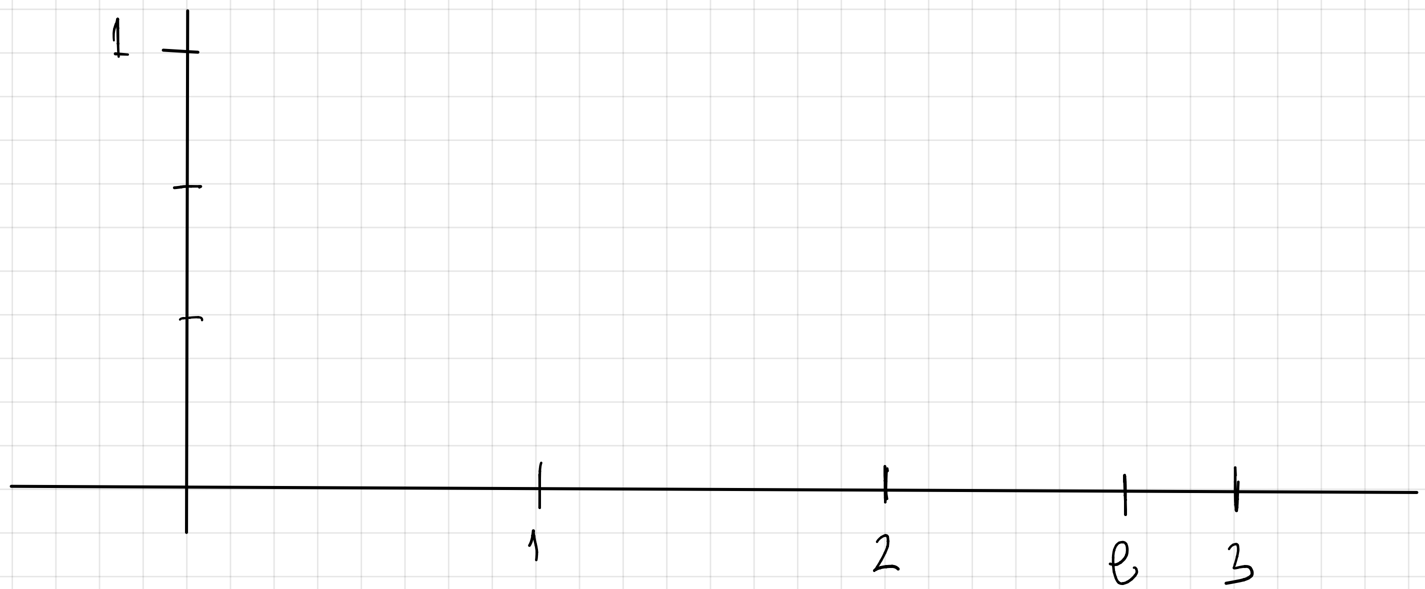
By the Radon measure theorem, Borel probability measures on $\mathbb{R}$ are characterized by their CDF. Note: for probability measures μ:

$$\lim_{x \to -\infty} F_\mu(x) =$$

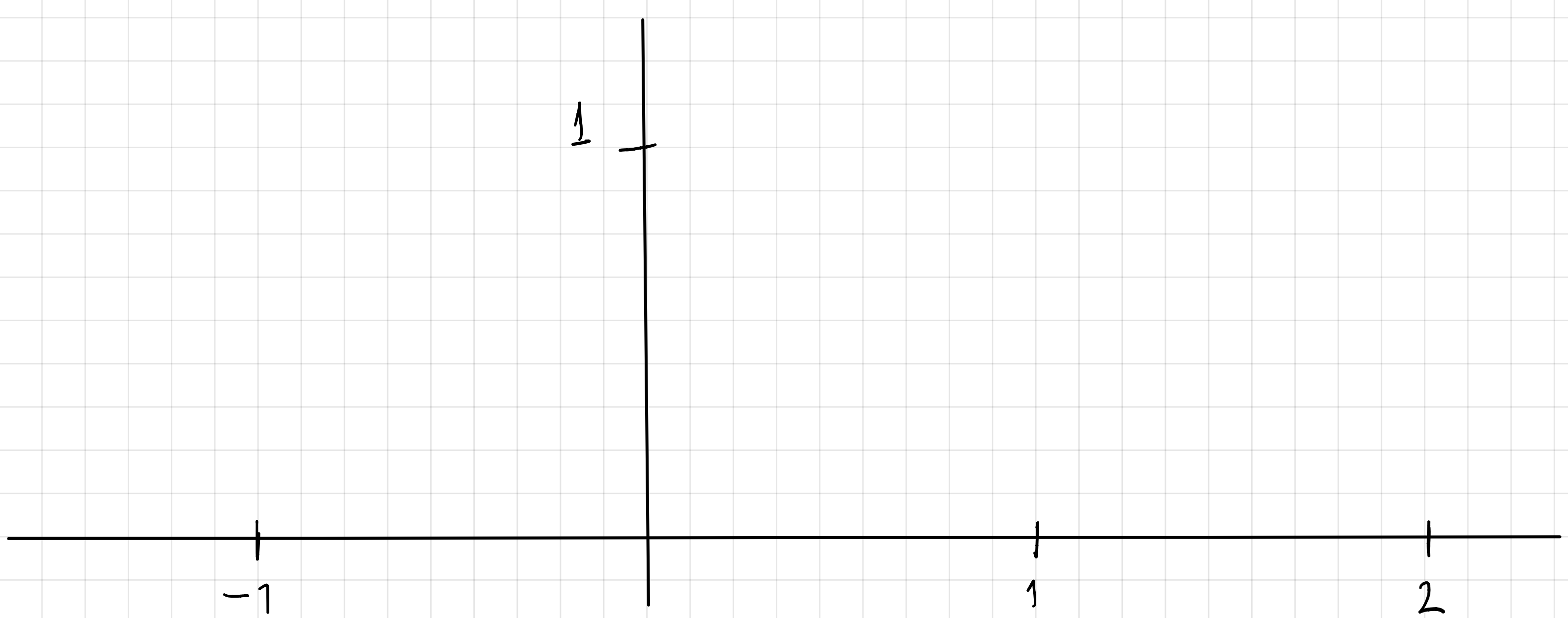
$$\lim_{x \to +\infty} F_\mu(x) =$$

Cor: Any right-continuous, non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$ is the CDF of a unique Borel probability measure on $\mathbb{R}$.

E.g. $\mu = \frac{1}{3}\delta_0 + \frac{1}{9}\delta_1 + \frac{5}{9}\delta_e$

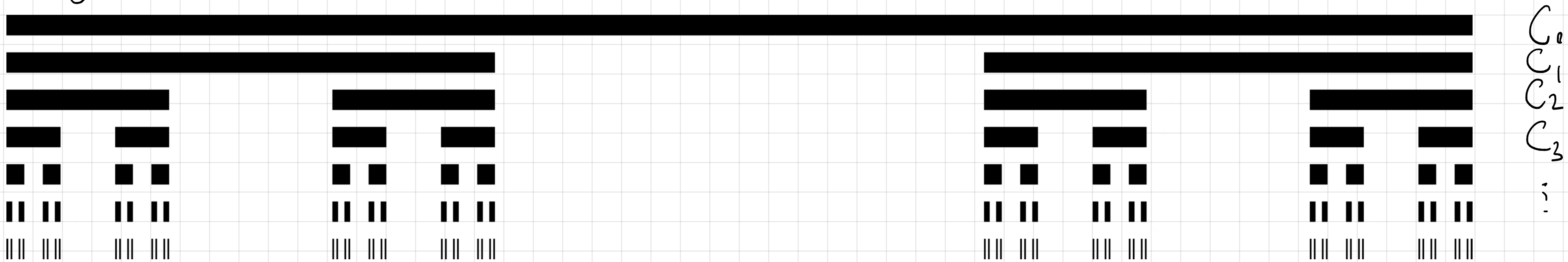


E.g. $F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$

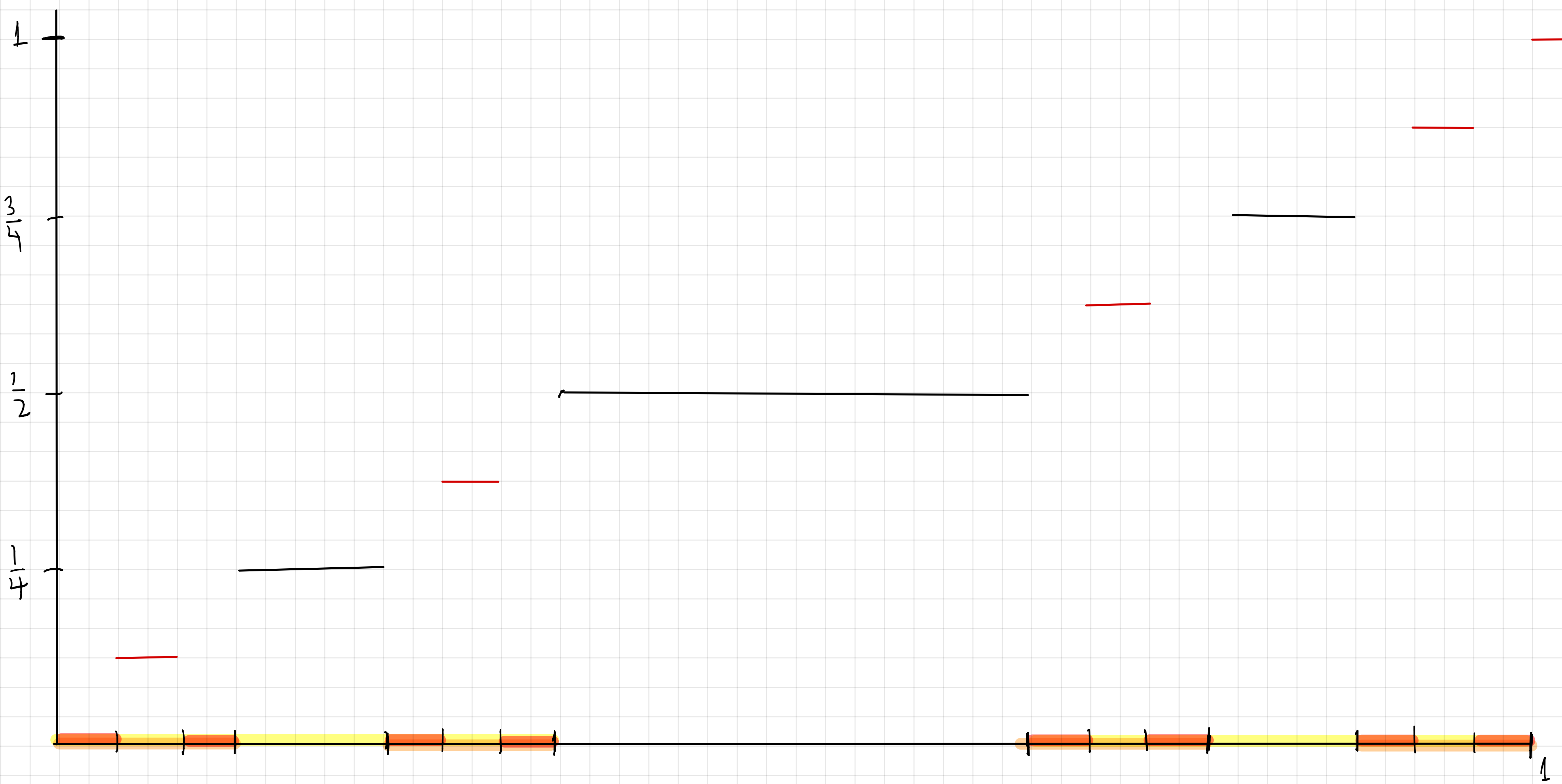


Eg. The devil's staircase.

Begin by constructing the Cantor set:



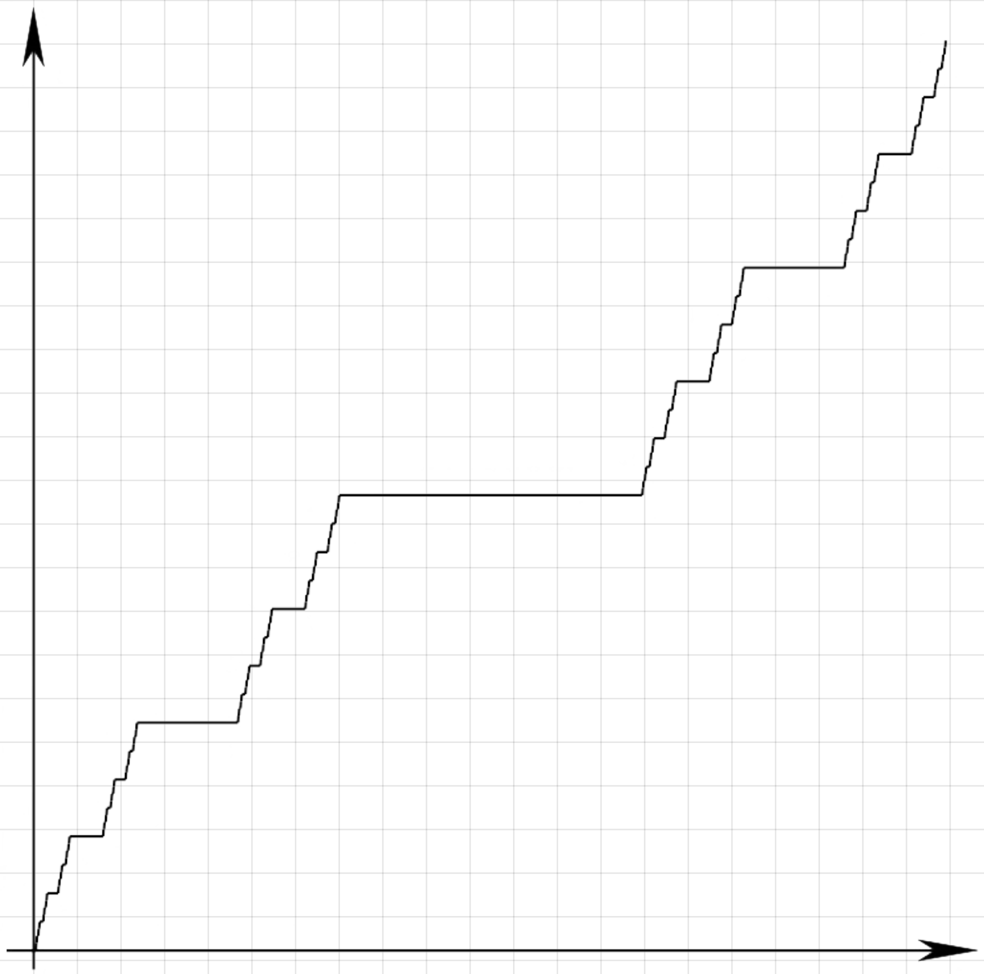
Now, let $F_n(x) = \int_0^x \left(\frac{3}{2}\right)^n \mathbb{1}_{C_n}(t)\,dt$.



Undergraduate Analysis Exercise:

$F_n(x) \longrightarrow F(x)$

continuous, non-decreasing function



- Continuous
- $F(x) = 0, \ x \le 0$
- $F(x) = 1, \ x \geq 1$
- F is non-decreasing

$\left.\right\}$ $\therefore \exists!$ Borel probability measure with CDF F.

μ_{Cantor}