If Ω is a topological space, any measure on $\mathcal{B}(\Omega)$ will be referred to as a Borel Measure.

We will principally work with Borel measures throughout this course.

Radon Measures (§ 6.5 in Driver)

A Borel measure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ on $\mathbb{R}$ is called a Radon measure if

$$\mu([a,b]) < \infty \quad \forall\, a<b \in \mathbb{R}.$$

(More generally, a Radon measure on a topological space Ω is a Borel measure μ s.t. $\mu(K) < \infty$ for all compact K (and satisfies some regularity conditions that turn out be be automatic when $\Omega = \mathbb{R}$).

Eg. The Stieltjes premeasures $\mu_F((a,b]) = F(b) - F(a)$ for $F: \mathbb{R} \to \mathbb{R}$ increasing and right continuous.

Theorem: If μ is a Radon measure on $\mathbb{R}$, then there exists a non-decreasing, right-continuous function $F:\mathbb{R}\to\mathbb{R}$ (unique up to an additive constant) s.t.

$$\mu((a,b]) = F(b)-F(a), \quad -\infty \le a < b \le \infty .$$

$$\| $$

$$G(b)-G(a)$$

Pf. Uniqueness:

$$\therefore\ G(b) = F(b) + \underbrace{[G(0)-F(0)]}_{\text{const.}} \qquad a=0$$

Existence: If $\mu(\mathbb{R})<\infty$, $F(x) = \mu(-\infty,x]$ (if $<\infty$)

/ If $\mu(B)<\infty$, $A\subseteq B$, $\mu(B\setminus A) = \mu(B)-\mu(A)$. /

$$\mu(a,b] = \mu((-\infty,b]\setminus(-\infty,a]) = \mu(-\infty,b]-\mu(-\infty,a] = F(b)-F(a).$$

If $x\le y$, $F(y) = \mu(\underset{(-\infty,x]}{\overset{\cup\!\mid}{(-\infty,y]}}) \ge \mu(-\infty,x] = F(x)$.

Right continuity: Let $x_n \downarrow x$. Then $(x_n, x_1] \subseteq (x_{n+1}, x_1]$

$$\bigcup_{n=1}^{\infty} (x_n, x_1] = (x, x_1].$$

$(x_n, x_1] \uparrow (x, x_1]$. $\leftarrow$ $\mu(x, x_1] < \infty$

$\therefore (x_n, x_1] \xrightarrow{d_\mu} (x, x_1]$.

$\therefore \mu((x, x_1] \setminus (x_n, x_1]) \to 0$.

$$\mu(x, x_1] - \mu(x_n, x_1]$$

$$(\cancel{F(x_1)} - F(x)) - (\cancel{F(x_1)} - F(x_n))$$

$$= F(x_n) - F(x) \to 0.$$

In general:

$$F(x) = \begin{cases} \mu(0, x] & x \geq 0 \\ -\mu(x, 0] & x \leq 0 \end{cases}$$

We saw (Lecture 3.1) that $(\Omega, \mathcal{B}_{(]}, \mu_F)$ is a premeasure for every right-continuous increasing $F: \mathbb{R} \to \mathbb{R}$. $\therefore$ By the extension theorem, we now have a characterization of Radon measures on $\mathbb{R}$.

Re: Convergence of $\mu(A_n)$, more generally: [Driver, Prop. 6.3]

Prop: Let μ be a finitely additive measure on $(\Omega, \mathcal{A})$. TFAE:

(1) μ is a premeasure on $\mathcal{A}$.

(2) If $A_n, A \in \mathcal{A}$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.

Moreover, in the case $\mu(\Omega) < \infty$, the following are also equivalent:

(3) If $A_n \downarrow A$ in $\mathcal{A}$, then $\mu(A_n) \downarrow \mu(A)$. $\quad A_n \supseteq A_{n+1}$, $A = \bigcap_n A_n$

(4) If $A_n \uparrow \Omega$ in $\mathcal{A}$, then $\mu(A_n) \uparrow \mu(\Omega)$.

(5) If $A_n \downarrow \emptyset$ in $\mathcal{A}$, then $\mu(A_n) \downarrow 0$.

Pf: (1) $\Rightarrow$ (2) If $A_n \uparrow A$, $B_n := A_n \setminus A_{n-1}$ disjoint, $\bigsqcup_{n=1}^{N} B_n = A_N$, $\bigsqcup_{n=1}^{\infty} B_n = A$.

$\therefore$ by (1) $\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) = \lim_{N\to\infty} \mu\left(\bigsqcup_{n=1}^{N} B_n\right) = \lim_{N\to\infty} \mu(A_N)$.

(2) $\Rightarrow$ (1) If $B = \bigsqcup_{n=1}^{\infty} B_n$, then $A_N = \bigsqcup_{n=1}^{N} B_n \uparrow B$. $\therefore \mu(B) = \lim_{N\to\infty} \mu\left(\bigsqcup_{n=1}^{N} B_n\right)$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) = \sum_{n=1}^{\infty} \mu(B_n)$$

To include (3-5), use the fact (true in the finite measure case) that

$$A \subseteq B \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$$

Def: Let μ be a Borel probability measure on $\mathbb{R}$.

$$F_\mu : \mathbb{R} \to \mathbb{R} \; ; \; F_\mu(x) = \mu((-\infty, x])$$

is the cumulative distribution function (CDF) of μ.

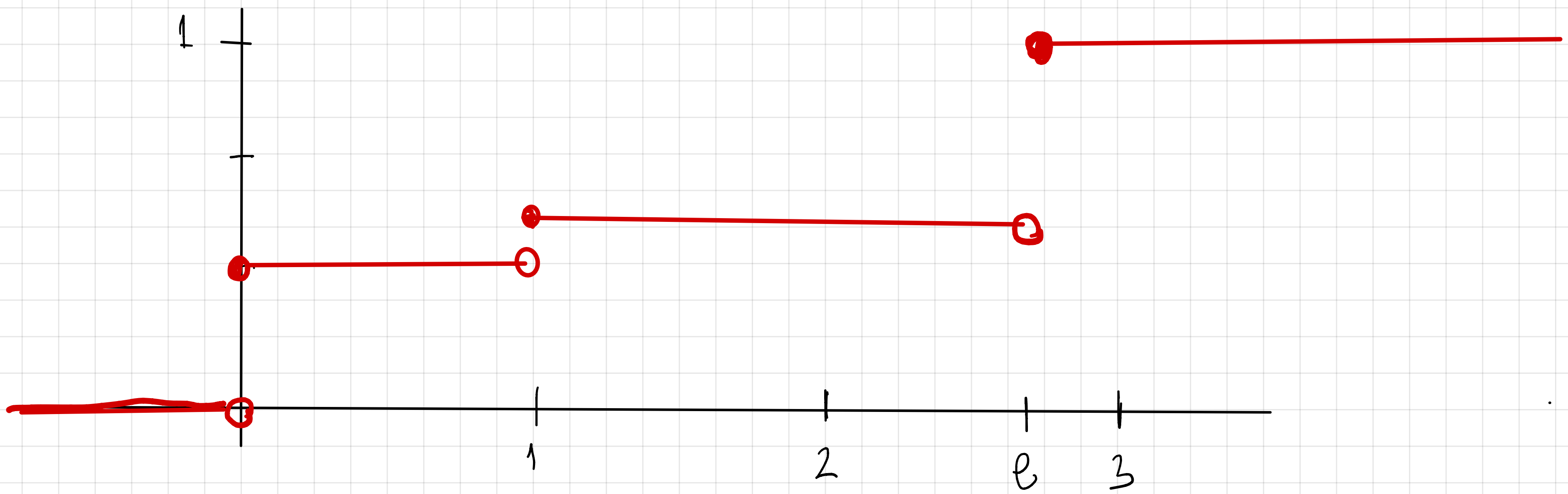
By the Radon measure theorem, Borel probability measures on $\mathbb{R}$ are characterized by their CDF. Note: for probability measures μ:

$$\lim_{x \to -\infty} F_\mu(x) = \lim_{x \to -\infty} \mu((-\infty, x]) = \mu(\emptyset) = 0.$$

$$\lim_{x \to +\infty} F_\mu(x) = \lim_{x \to +\infty} \mu((-\infty, x]) = \mu(\mathbb{R}) = 1.$$

Cor: Any right-continuous, non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$ is the CDF of a unique Borel probability measure on $\mathbb{R}$.

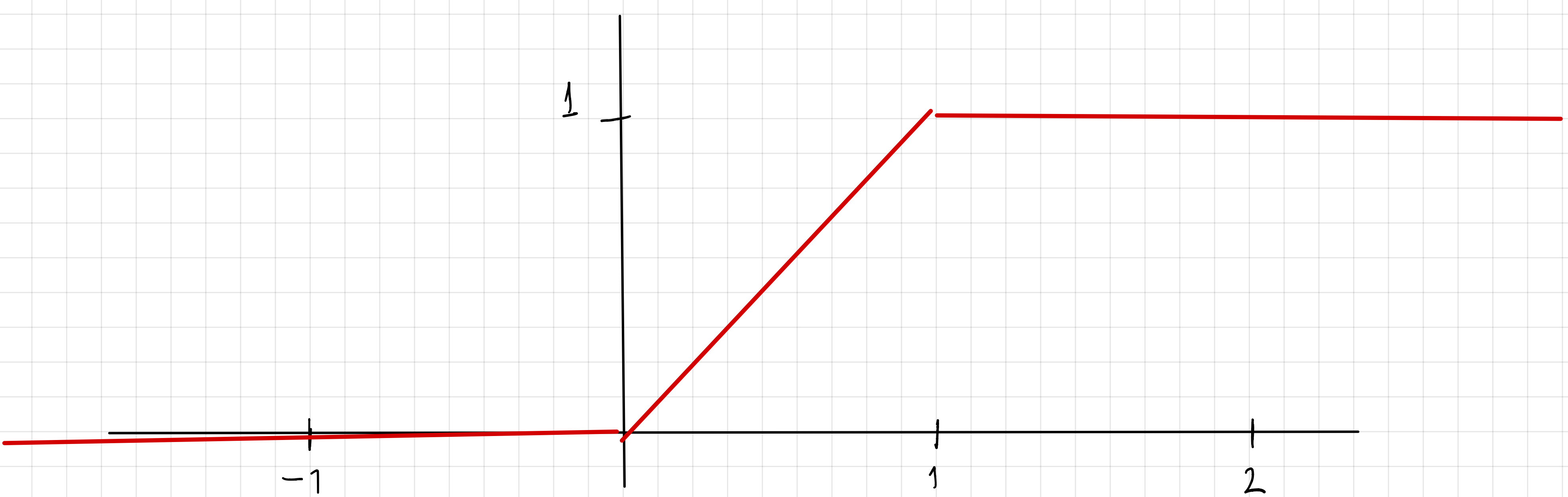
E.g. $\mu = \frac{1}{3}\delta_0 + \frac{1}{9}\delta_1 + \frac{5}{9}\delta_e$ $\qquad F_\mu(x) = \mu((-\infty, x])$



Pure point mass/ purely discrete

∴ CDF is a step function.

E.g. $F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$ continuous

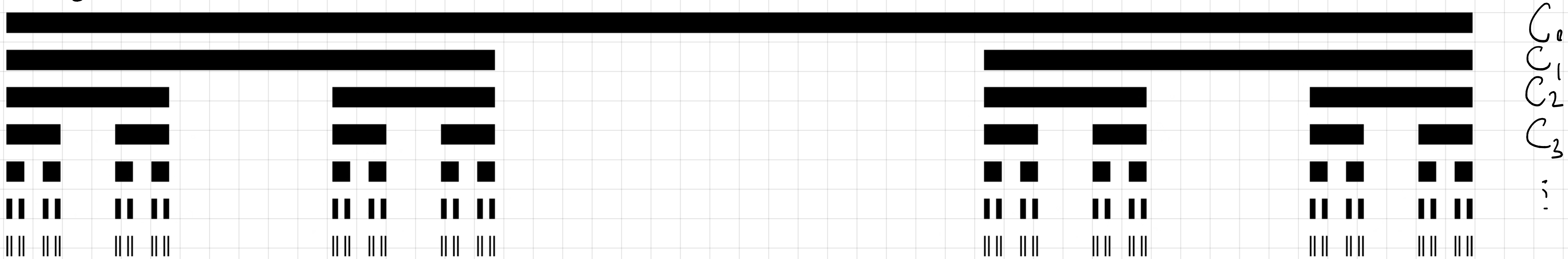


If $x_0 \in \mathbb{R}$,

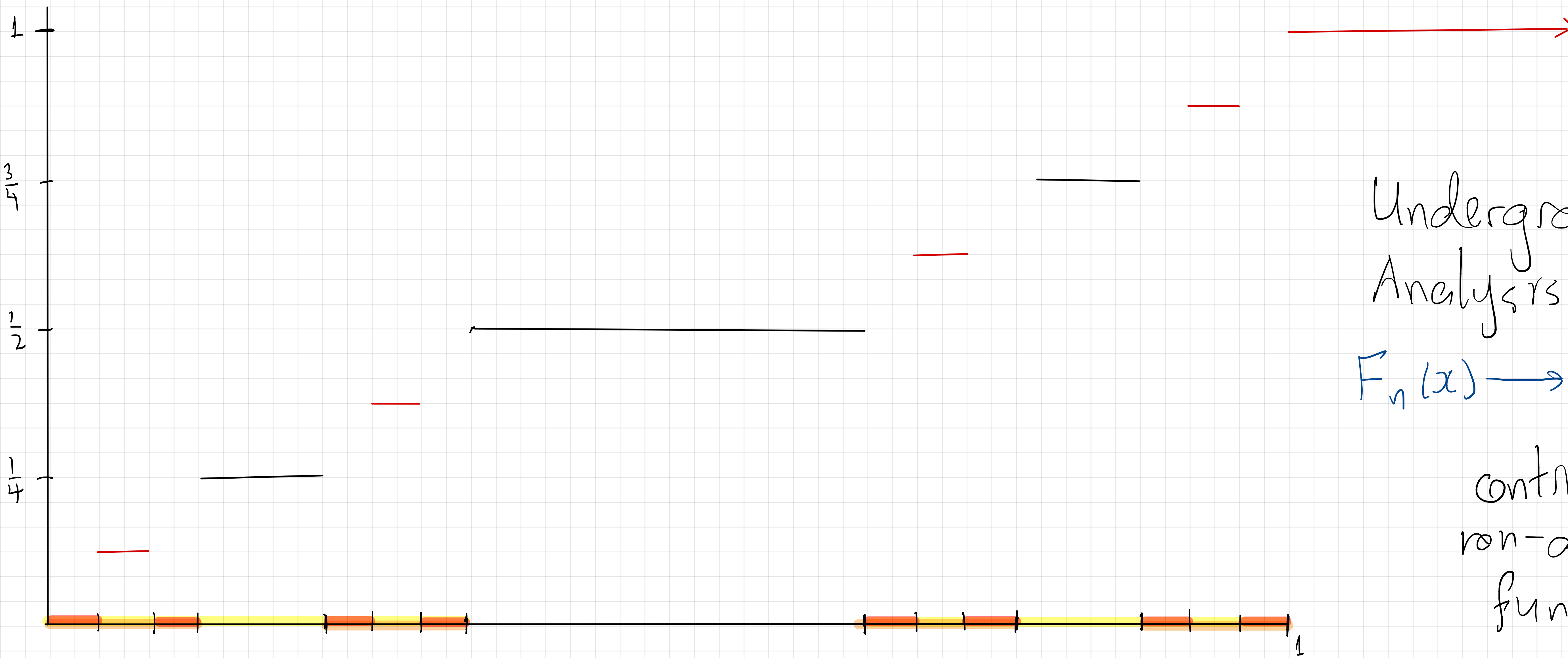
$$\mu(\{x_0\}) = \lim_{x \to x_0} \mu(x, x_0] = \lim_{x \to x_0} (F(x_0) - F(x)) = 0$$

E.g. The devil's staircase.

Begin by constructing the Cantor set:



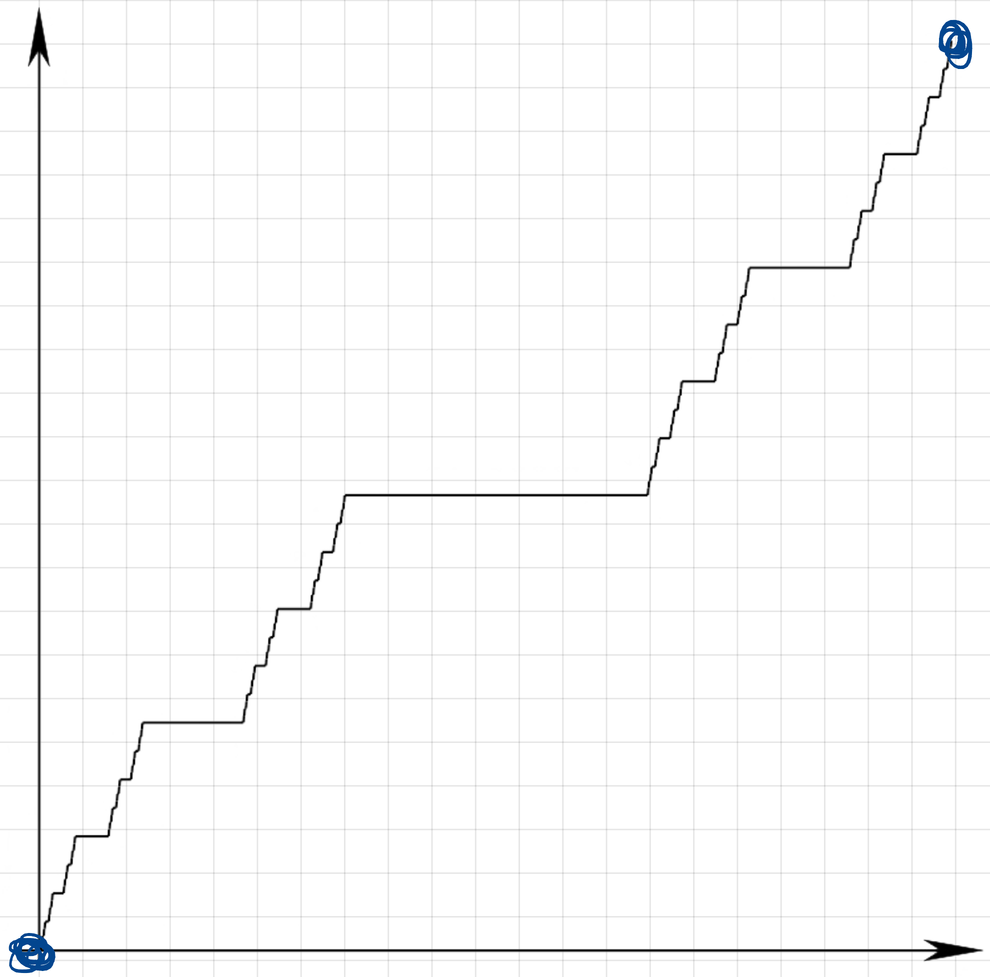
Now, let $F_n(x) = \int_0^x \left(\frac{3}{2}\right)^n \mathbb{1}_{C_n}(t)\, dt$.



Undergraduate Analysis Exercise:

$F_n(x) \longrightarrow F(x)$

continuous, non-decreasing function



- Continuous
- $F(x)=0, \; x \leq 0$
- $F(x)=1, \; x \geq 1$
- F is non-decreasing

$\therefore \exists!$ Borel probability measure with CDF F.

μ_{Cantor}

$\mu_{\text{Cantor}}(\{x_0\}) = 0 \quad \forall x_0.$

$F'(x) = 0$ "almost everywhere"

"singular continuous".