Extension Theorem Review (Driver's Approach; see also Maharam, 1987)

1. $(\Omega, \mathcal{A}, \mu)$ finite premeasure space

   $\mu^* : 2^\Omega \to [0, \mu(\Omega)] : \mu^*(E) = \inf\{\sum_{j=1}^\infty \mu(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^\infty A_j\}$

   ↳ monotone, countably subadditive

   ↳ If ν is a measure on $\mathcal{F} \supseteq \mathcal{A}$ extending μ, then $\nu \le \mu^*$ on $\mathcal{F}$.

   ↳ If χ is a finitely additive measure, $\chi^* \le \chi$ and $\chi = \chi^*$ iff χ is countably additive.

2. Outer pseudo-metric $d_\mu : 2^\Omega \times 2^\Omega \to [0, \mu(\Omega)] : d_\mu(E, F) = \mu^*(E \Delta F)$

   ↳ Is a pseudo-metric!

   ↳ well behaved w.r.t. unions, intersections, complements

3. $\mu : \mathcal{A} \to \mathbb{R}$ is Lip-1 on the pseudo-metric space, so extends uniquely to $\bar{\mu} : \bar{\mathcal{A}} \to \mathbb{R}$.

4. $\mathcal{A}_\sigma = \{$countable unions of sets in $\mathcal{A}\}$

   ↳ In the pseudo-metric space $(2^\Omega, d_\mu)$, $\overline{\mathcal{A}_\sigma} = \bar{\mathcal{A}}$

   ↳ $\bar{\mu} = \mu^*$ on $\mathcal{A}_\sigma$.

5. $\bar{\mathcal{A}}$ is a σ-field.

6. Outer approximation of $\bar{\mathcal{A}}$ by $\mathcal{A}_\sigma$:

$$B \in \bar{\mathcal{A}} \iff \forall \varepsilon > 0\ \exists C \in \mathcal{A}_\sigma \text{ s.t. } B \subseteq C \text{ and } \mu^*(C \setminus B) = d_\mu(B, C) < \varepsilon.$$

↳ use this, together with $\mu^* = \bar{\mu}$ on $\mathcal{A}_\sigma$, to show $\mu^* = \bar{\mu}$ on $\bar{\mathcal{A}}$.

7. $\bar{\mu}$ is a countably additive measure on $\bar{\mathcal{A}} \supseteq \sigma(\mathcal{A})$.

8. Uniqueness Theorem: If $\mathcal{F}$ is a σ-field with $\mathcal{A} \subseteq \mathcal{F} \subseteq \bar{\mathcal{A}}$, and ν is a measure on $\mathcal{F}$ with $\nu|_{\mathcal{A}} = \mu$, the $\nu = \bar{\mu}|_{\mathcal{F}}$.

Pf.

Extension to σ-Finite Measures

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite premeasure space

$\Omega = \bigcup_{n=1}^{\infty} A_n$ s.t. $A_n \in \mathcal{A}$, $\mu(A_n) < \infty$.

Take $\Omega_1 = A_1$, $\Omega_n = A_n \setminus A_{n-1}$ so $\mu(\Omega_n) \leq \mu(A_n) < \infty$, $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$.

Define $\mu_n : \mathcal{A} \to [0, \infty) : \mu_n(A) = \mu(A \cap \Omega_n)$

Then $(\Omega_n, \mathcal{A}, \mu_n)$ is a finite premeasure space

↳ Extend to a finite measure $\bar{\mu}_n$ on $\bar{\mathcal{A}}^n$

Theorem: $\bar{\mu} := \sum_{n=1}^{\infty} \bar{\mu}_n$ is the unique measure on $\sigma(\mathcal{A})$ extending μ.

Pf. Easy to check that $\bar{\mu}$ is a countably-additive measure (b/c the Ω_n are disjoint). We need to check uniqueness.

Suppose ν is a measure on $\sigma(\mathcal{A})$ s.t. $\nu|_{\mathcal{A}} = \mu$.

Proposition: $(\Omega, \mathcal{A}, \mu)$ σ-finite premeasure space.

1. $\bar{\mu} = \mu^*$ on $\sigma(\mathcal{A})$.
2. If $B \in \sigma(\mathcal{A})$ and $\varepsilon > 0$, $\exists\, C \in \mathcal{A}_\sigma$ s.t. $B \subseteq C$ and $\bar{\mu}(C \setminus B) < \varepsilon$.
3. Moreover, if $\bar{\mu}(B) < \infty$, $\exists\, A \in \mathcal{A}$ s.t. $\bar{\mu}(A \Delta B) < \varepsilon$.

Pf. 1. & 2. follow by "$\varepsilon/2^n$"-style extension arguments.
see [Driver, Cor. 6.29 - 6.30].
Let's focus on the second statement.