# Extension Theorem Review (Driver's Approach; see also Maharam, 1987)

1. $(\Omega, \mathcal{A}, \mu)$ finite premeasure space

   $\mu^*: 2^\Omega \to [0, \mu(\Omega)] : \mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^{\infty} A_j\right\}$

   ↳ monotone, countably subadditive

   ↳ If $\nu$ is a measure on $\mathcal{F} \supseteq \mathcal{A}$ extending $\mu$, then $\nu \leq \mu^*$ on $\mathcal{F}$. ☆

   ↳ If $\chi$ is a finitely additive measure, $\chi^* \leq \chi$ and $\chi = \chi^*$ iff $\chi$ is countably additive.

2. Outer pseudo-metric $d_\mu : 2^\Omega \times 2^\Omega \to [0, \mu(\Omega)] : d_\mu(E, F) = \mu^*(E \Delta F)$

   ↳ Is a pseudo-metric!

   ↳ well behaved w.r.t. unions, intersections, complements

3. $\mu : \mathcal{A} \to \mathbb{R}$ is Lip-1 on the pseudo-metric space, so extends uniquely to $\bar{\mu} : \bar{\mathcal{A}} \to \mathbb{R}$.

4. $\mathcal{A}_\sigma = \{$countable unions of sets in $\mathcal{A}\}$

   ↳ In the pseudo-metric space $(2^\Omega, d_\mu)$, $\overline{\mathcal{A}_\sigma} = \bar{\mathcal{A}}$

   ↳ $\bar{\mu} = \mu^*$ on $\mathcal{A}_\sigma$.

5. $\bar{\mathcal{A}}$ is a $\sigma$-field.

6. Outer approximation of $\bar{\mathcal{A}}$ by $\mathcal{A}_\sigma$:

$$B \in \bar{\mathcal{A}} \iff \forall \varepsilon > 0 \ \exists C \in \mathcal{A}_\sigma \text{ s.t. } B \subseteq C \text{ and } \mu^*(C \setminus B) = d_\mu(B, C) < \varepsilon .$$

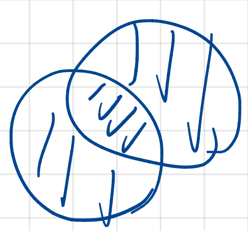
↳ use this, together with $\mu^* = \bar{\mu}$ on $\mathcal{A}_\sigma$, to show $\mu^* = \bar{\mu}$ on $\bar{\mathcal{A}}$.

7. $\bar{\mu}$ is a countably additive measure on $\bar{\mathcal{A}} \supseteq \sigma(\mathcal{A})$.

8. <u>Uniqueness</u> <u>Theorem</u>: If $\mathcal{F}$ is a $\sigma$-field with $\mathcal{A} \subseteq \mathcal{F} \subseteq \bar{\mathcal{A}}$, and $\nu$ is a measure on $\mathcal{F}$ with $\nu|_{\mathcal{A}} = \mu$, the $\nu = \bar{\mu}|_{\mathcal{F}}$.

<u>Pf.</u> $\nu \leq \mu^*$ on $\mathcal{F}$.

Let $A, B \in \mathcal{F}$ $\quad |\nu(B) - \nu(A)| \leq \nu(A \triangle B) \leq \mu^*(A \triangle B) = d_\mu(A, B)$



$\nu : \mathcal{F} \to \mathbb{R}$ is Lip-1 wrt $d_\mu$.

$E$, $\exists A_n \in \mathcal{A}$ s.t. $A_n \xrightarrow{d_\mu} E$.

$$\therefore \ \nu(A_n) \longrightarrow \nu(E)$$
$$\mu(A_n) \longrightarrow \bar{\mu}(E) \qquad ///$$

# Extension to $\sigma$-Finite Measures

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite premeasure space

$\Omega = \bigcup_{n=1}^{\infty} A_n$ s.t. $A_n \in \mathcal{A}$, $\mu(A_n) < \infty$.

Take $\Omega_1 = A_1$, $\Omega_n = A_n \setminus A_{n-1}$ so $\mu(\Omega_n) \leq \mu(A_n) < \infty$, $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$.

Define $\mu_n : \mathcal{A} \to [0, \infty) : \mu_n(A) = \mu(A \cap \Omega_n)$

Then $(\Omega_n, \mathcal{A}, \mu_n)$ is a finite premeasure space

↳ Extend to a finite measure $\bar{\mu}_n$ on $\bar{\mathcal{A}}^n \supseteq \sigma(\mathcal{A})$

**Theorem:** $\bar{\mu} := \sum_{n=1}^{\infty} \bar{\mu}_n$ is the unique measure on $\sigma(\mathcal{A})$ extending $\mu$.

**Pf.** Easy to check that $\bar{\mu}$ is a countably-additive measure (b/c the $\Omega_n$ are disjoint). We need to check uniqueness.

Suppose $\nu$ is a measure on $\sigma(\mathcal{A})$ s.t. $\nu|_{\mathcal{A}} = \mu$.

Define $\nu_n$ on $\sigma(\mathcal{A})$ as $\nu_n(E) = \nu(E \cap \Omega_n)$

For $A \in \mathcal{A}$, $\nu_n(A) = \nu(\underset{\substack{\cap \\ \mathcal{A}}}{A \cap \Omega_n}) = \mu(A \cap \Omega_n) = \mu_n(A)$.

$\therefore$ $\nu_n$ is a finite measure extending $\mu_n$

$\therefore$ by uniqueness in the finite case, $\nu_n = \bar{\mu}_n|_{\sigma(\mathcal{A})}$.

$\therefore$ For any $E \in \sigma(\mathcal{A})$, $E = \bigsqcup_{n=1}^{\infty} E \cap \Omega_n$

$$\therefore \nu(E) = \sum_{n=1}^{\infty} \nu(E \cap \Omega_n)$$

$$= \sum_{n=1}^{\infty} \nu_n(E) = \sum_{n=1}^{\infty} \bar{\mu}_n(E) = \bar{\mu}(E).$$

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**Proposition:** $(\Omega, \mathcal{A}, \mu)$ $\sigma$-finite premeasure space.

1. $\bar{\mu} = \mu^*$ on $\sigma(\mathcal{A})$.
2. If $B \in \sigma(\mathcal{A})$ and $\varepsilon > 0$, $\exists\, C \in \mathcal{A}_\sigma$ s.t. $B \subseteq C$ and $\bar{\mu}(C \setminus B) < \varepsilon$.
3. Moreover, if $\bar{\mu}(B) < \infty$, $\exists\, A \in \mathcal{A}$ s.t. $\bar{\mu}(A \triangle B) < \varepsilon$.

**Pf.** 1. & 2. follow by "$\varepsilon/2^n$"-style extension arguments.

see [Driver, Cor. 6.29 - 6.30].

Let's focus on the second statement.

Find $C \in \mathcal{A}_\sigma$ s.t. $B \subseteq C$, $\bar{\mu}(C \setminus B) < \varepsilon/2$.

Find $\underset{\mathcal{A}}{\underset{\cap}{A_n}} \uparrow C$.

$$\bar{\mu}(A_n \triangle B) = \bar{\mu}(A_n \setminus B) + \bar{\mu}(B \setminus A_n)$$

$A_n \setminus B \sqcup B \setminus A_n$

$$\leq \underbrace{\bar{\mu}(C \setminus B)}_{\varepsilon/2} + \bar{\mu}(B \setminus A_n)$$

$$\bar{\mu}(B \setminus A_n) \leq \bar{\mu}(C \setminus A_n) = \bar{\mu}(C) - \mu(A_n) \to 0 \text{ as } n \to \infty$$

$\therefore \exists\, n_0 < \infty$, $\varepsilon/2$

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Note: $\bar{\mu}(C) < \infty$

$C \subseteq C \setminus B \cup B$

$$\bar{\mu}(C) \leq \bar{\mu}(C \setminus B) + \bar{\mu}(B) < \frac{\varepsilon}{2} + \bar{\mu}(B) < \infty$$