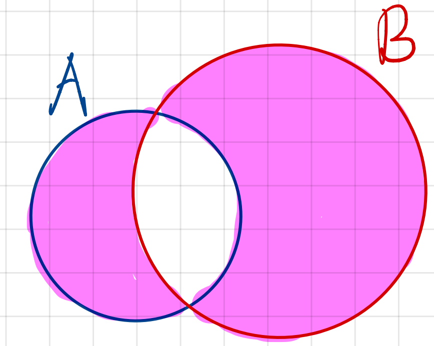
Outer Pseudo-Metric Closure (§6.2 in Driver)

- $(\Omega, \mathcal{A}, \mu)$ finite premeasure space
- $\mu^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\} \quad \forall E \in 2^{\Omega}$
- $d_\mu(E,F) = \mu^*(E \Delta F)$

Theorem: The closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ in the pseudo-metric space $(2^{\Omega}, d_\mu)$ is a σ-field.

Now, we've proved that $\mu^*|_{\mathcal{A}} = \mu$. So, for $A, B \in \mathcal{A}$,

$d_\mu(A,B)$



Prop: μ extends to a unique Lip-1 function $\bar{\mu} : \bar{\mathcal{A}} \to [0, \mu(\Omega)]$.

Def: Given $\mathcal{E} \subseteq 2^{\Omega}$, $\mathcal{E}_\sigma := \{$countable unions of elements of $\mathcal{E}\}$

↳ Note: $\mathcal{E}_\sigma$ is automatically closed under countable unions.

If $\mathcal{E}$ is closed under finite intersections, so is $\mathcal{E}_\sigma$:

Restatement of Lemma (from last time):

If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then $\overline{\mathcal{A}_\sigma} = \overline{\mathcal{A}}$, and $\overline{\mu} = \mu^*$ on $\mathcal{A}_\sigma$.

Pf. We showed that if $\mathcal{A} \ni A_n \uparrow A$ then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \xrightarrow[n\to\infty]{} 0$.

Prop: Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. For $B \in 2^{\Omega}$, TFAE:

(1) $B \in \overline{\mathcal{A}}$.

(2) $\forall \varepsilon > 0$, $\exists C \in \mathcal{A}_\sigma$ s.t. $B \subseteq C$ and $\mu^*(C \setminus B) = d_\mu(B, C) < \varepsilon$.

Pf. (2) $\Rightarrow$ (1): Select a sequence $C_n \in \mathcal{A}_\sigma$ s.t. $d_\mu(B, C_n) < \frac{1}{n}$; then $C_n \to B$ and so $B \in \overline{\mathcal{A}_\sigma}$

(1) $\Rightarrow$ (2):

Cor: Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space.
Then $\mu^* = \bar{\mu}$ on $\bar{\mathcal{A}}$.

Pf. Let $B \in \bar{\mathcal{A}}$. $\bar{\mu}(B) =$

Theorem: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then $\bar{\mu}: \bar{\mathcal{A}} \to [0, \mu(\Omega)]$ is a measure.

Pf. - We will show that $\bar{\mu}$ is finitely-additive on $\bar{\mathcal{A}}$. Once we've done that: we've shown $\bar{\mu}$ is a finitely-additive measure on the σ-field $\bar{\mathcal{A}}$, and $\therefore$ it is countably super-additive. But by the prev. Corollary, $\bar{\mu} = \mu^*$ on $\bar{\mathcal{A}}$, and μ^* is countably subadditive. ✓