Outer Pseudo-Metric Closure (§6.2 in Driver)

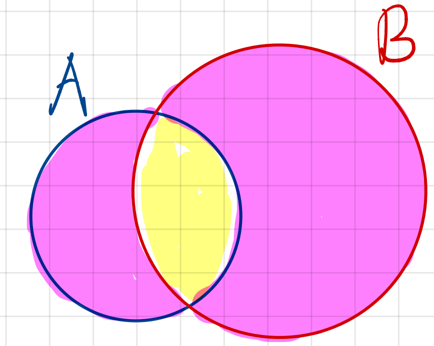
$\mu(A\triangle B)+\mu(A\cap B) = \mu(A\cup B)$

- $(\Omega,\mathcal{A},\mu)$ finite premeasure space
- $\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty}\mu(A_j) : A_j\in\mathcal{A},\ E\subseteq\bigcup_{j=1}^{\infty}A_j\right\}\quad\forall E\in 2^{\Omega}$
- $d_\mu(E,F) = \mu^*(E\triangle F)$

Theorem: The closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in the pseudo-metric space $(2^{\Omega}, d_\mu)$ is a σ-field.

Now, we've proved that $\mu^*|_{\mathcal{A}} = \mu$. So, for $A,B\in\mathcal{A}$,

$$d_\mu(A,B) = \mu^*(\underbrace{A\triangle B}_{\in\mathcal{A}}) = \mu(A\triangle B)$$
$$= \mu(A\cup B) - \mu(A\cap B) \geq |\mu(A)-\mu(B)|$$
$$\mu(B) - \mu(A)$$
$$\mu(A) - \mu(B)$$



$\therefore\ \mu\in \text{Lip-}1$ on $\mathcal{A}$

Prop: μ extends to a unique Lip-1 function $\bar{\mu}:\overline{\mathcal{A}}\to[0,\mu(\Omega)]$.

Def: Given $\mathcal{E} \subseteq 2^{\Omega}$, $\mathcal{E}_\sigma := \{$countable unions of elements of $\mathcal{E}\}$

$\hookrightarrow$ Note: $\mathcal{E}_\sigma$ is automatically closed under countable unions.

If $\mathcal{E}$ is closed under finite intersections, so is $\mathcal{E}_\sigma$:

$$\left(\bigcup_{n=1}^{\infty} E_i\right) \cap \left(\bigcup_{j=1}^{\infty} F_j\right) = \bigcup_{i,j=1}^{\infty} \underbrace{E_i \cap F_j}_{\in \mathcal{E}}$$

Restatement of Lemma (from last time):

If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then $\overline{\mathcal{A}_\sigma} = \overline{\mathcal{A}}$, and $\overline{\mu} = \mu^*$ on $\mathcal{A}_\sigma$.

Pf. We showed that if $\mathcal{A} \ni A_n \uparrow A$ then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \xrightarrow[n\to\infty]{} 0$.

If $A \in \mathcal{A}_\sigma$. $\therefore A = \bigcup_{n=1}^{\infty} \underset{\mathcal{A}}{B_n}$

Define $A_n = \bigcup_{j=1}^{n} B_j$

$\Rightarrow A_n \uparrow A$

$\therefore d_\mu(A_n, A) \to 0$

$\therefore \mathcal{A}_\sigma \ni A \in \overline{\mathcal{A}}$.

$\overline{\mathcal{A}} \subseteq \overline{\mathcal{A}_\sigma} \subseteq \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$

$\Rightarrow \overline{\mathcal{A}_\sigma} = \overline{\mathcal{A}}$

$\mu^*(A) = \lim_{n\to\infty} \mu(A_n) = \overline{\mu}(A)$. //

Prop: Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. For $B \in 2^{\Omega}$, TFAE:

(1) $B \in \overline{\mathcal{A}}$.

(2) $\forall \varepsilon > 0$, $\exists C \in \mathcal{A}_\sigma$ s.t. $B \subseteq C$ and $\mu^*(C \setminus B) = d_\mu(B, C) < \varepsilon$.

Pf. $(2) \not\Rightarrow (1)$: Select a sequence $C_n \in \mathcal{A}_\sigma$ s.t. $d_\mu(B, C_n) < \frac{1}{n}$; then $C_n \to B$ and so $B \in \overline{\mathcal{A}_\sigma} = \overline{\mathcal{A}}$.

$(1) \Rightarrow (2)$: Fix $\varepsilon > 0$. By (1), find $G \in \mathcal{A}$ s.t. $\varepsilon > d_\mu(B, G) = \mu^*(B \triangle G)$

$$\varepsilon > \mu^*(B \triangle G) = \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A},\ B \triangle G \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

$\therefore$ we can find $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ s.t.

$$B \triangle G \subseteq \bigcup_{n=1}^{\infty} A_n, \quad \sum_{n=1}^{\infty} \mu(A_n) < \varepsilon$$

Set $C := G \cup \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\sigma$. $\therefore B \subseteq (B \setminus G) \cup G \subseteq (B \triangle G) \cup G \subseteq C$

and $C \setminus B = (G \setminus B) \cup \left(\bigcup_{n=1}^{\infty} A_n \setminus B \right) \subseteq (B \triangle G) \cup \left(\bigcup_{n=1}^{\infty} A_n \setminus B \right) \subseteq \bigcup_{n=1}^{\infty} A_n$

$$\therefore \mu^*(C \setminus B) \leq \sum_{n=1}^{\infty} \mu(A_n) < \varepsilon.$$

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Cor: Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. Then $\mu^* = \bar{\mu}$ on $\bar{\mathcal{A}}$.

Pf. Let $B \in \bar{\mathcal{A}}$. $\bar{\mu}(B) = |\bar{\mu}(B) - \bar{\mu}(\emptyset)| \leq d_\mu(B, \emptyset) = \mu^*(B \triangle \emptyset) = \mu^*(B)$

For reverse ineq: Fix $\varepsilon > 0$. Choose $C \in \mathcal{A}_\sigma$ s.t. $B \subseteq C$ and

$$d_\mu(B, C) = \mu^*(C \setminus B) < \varepsilon.$$

$$|\bar{\mu}(B) - \bar{\mu}(C)| \leq d_\mu(B, C) < \varepsilon$$

$$\bar{\mu}(C) \leq \bar{\mu}(B) + \varepsilon$$

$$\bar{\mu}(B) \leq \mu^*(B) \leq \mu^*(C) = \bar{\mu}(C) \leq \bar{\mu}(B) + \varepsilon$$

Take $\varepsilon \searrow 0$: $\bar{\mu}(B) = \mu^*(B)$ ///

Theorem: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then $\bar{\mu} : \bar{\mathcal{A}} \to [0, \mu(\Omega)]$ is a measure.

Pf. We will show that $\bar{\mu}$ is finitely-additive on $\bar{\mathcal{A}}$. Once we've done that: we've shown $\bar{\mu}$ is a finitely-additive measure on the σ-field $\bar{\mathcal{A}}$, and $\therefore$ it is countably super-additive. But by the prev. Corollary, $\bar{\mu} = \mu^*$ on $\bar{\mathcal{A}}$, and μ^* is countably subadditive. ✓

Let $A, B \in \bar{\mathcal{A}}$. $\therefore$ Find $A_n \to A$, $B_n \to B$, $A_n, B_n \in \mathcal{A}$.

$$\left.\begin{array}{l} d_\mu(A_n \cup B_n, A \cup B) \\ d_\mu(A_n \cap B_n, A \cap B) \end{array}\right\} \le d_\mu(A_n, A) + d_\mu(B_n, B) \longrightarrow 0.$$

$$\therefore \bar{\mu}(A \cup B) + \bar{\mu}(A \cap B) = \lim_{n \to \infty} [\mu(A_n \cup B_n) + \mu(A_n \cap B_n)]$$
$$= \lim_{n \to \infty} [\mu(A_n) + \mu(B_n)] = \bar{\mu}(A) + \bar{\mu}(B).$$

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