Measure Extension Theorem

(Ω, A, μ) premeasure space.

$\mu^* : 2^\Omega \to [0, \infty]$ Carathéodory outer measure

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in A, \ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

Theorem: (Fréchet, Carathéodory, Hopf, Kolmogorov, ... 1920s)

There is a σ-field $M \supseteq A$ s.t. $\mu^*|_M$ is a measure.

Standard approach: $M = \{ E \subseteq \Omega : \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c), \ \forall T \subseteq \Omega \}$

- Show it is a σ-field, containing A
- Show μ^* is countably additive on it

Requires new tool:
Monotone Class Theorem
or
Dynkin's π-λ Theorem

Advantage: works for **all** premeasures.

Disadvantage: finicky, technical, unmotivated : too clever by half.

Driver's Approach   restrict to *finite* premeasures.

(Ω, A, μ) $\mu : A \to [0, \infty)$

$\quad\quad\quad\quad\quad\quad\quad$ in

Want extension: $\bar{\mu} : \bar{A} \nearrow$

1. Make 2^{Ω} into a topological space.
2. Define $\bar{A}$ to be the closure of A.
3. Prove $\mu : A \to [0, \infty)$ is sufficiently continuous,
   $\therefore$ extends to closure.
4. Use topological tools to show $\bar{A}$ is a σ-field,
   and $\bar{\mu}$ is a measure.

It will turn out that $\bar{\mu} = \mu^*$ on $\bar{A}$

Metric Spaces

$$d: X \times X \rightarrow [0, \infty)$$

1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$

- A sequence $(x_n)_{n=1}^{\infty}$ in X has a limit x if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; d(x_n, x) < \varepsilon.$$

- Given $V \subseteq X$, the closure $\overline{V}$ is the set of limits of sequences in V.

- A set V is closed if $\overline{V} = V$.

- A function $f: V \rightarrow \mathbb{R}$ is Lipschitz if $\exists K \in (0, \infty)$ s.t. $|f(x) - f(y)| \leq K d(x,y)$.

Prop: If f is Lipschitz on a nonempty $V \subseteq X$, then there is a unique Lipschitz extension $\bar{f}: \overline{V} \rightarrow \mathbb{R}$ (with the same Lipschitz constant K).
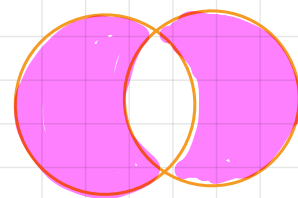
$$\bar{f}|_V = f.$$

The Outer Pseudo-Metric

(Ω, A, μ) <mark>finite</mark> premeasure space.

$\mu^* : 2^\Omega \to [0, \mu(\Omega)]$ Carathéodory outer measure

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in A, \ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

Def: $d_\mu : 2^\Omega \times 2^\Omega \to [0, \mu(\Omega)]$

$$d_\mu (E, F) = \mu^*(E \triangle F)$$

Prop: d_μ is a pseudo-metric on 2^Ω.

Pf:

Key Properties of the Outer Pseudo-Metric

1. $\forall\, A, B \in 2^{\Omega} \qquad d_\mu(A, B) = d_\mu(A^c, B^c)$.

2. $\forall\, \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in 2^{\Omega}$

   (a) $\quad d_\mu\left(\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n \right) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$

   (b) $\quad d_\mu\left(\bigcap_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} B_n \right) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$.

Pf.

Lemma: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, and $A_n \in \mathcal{A}$ with $A_n \uparrow A$, then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \to 0$.

Pf. Let $D_n = A_n \setminus A_{n-1}$. Then $A = \bigsqcup_{n=1}^{\infty} D_n$, and by definition of μ^*

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(D_n)$$

Cor: If $A_n \in \mathcal{A}$ and $A_n \uparrow A$, then $A \in \overline{\mathcal{A}}$.

Theorem: If (Ω, A, μ) is a finite premeasure space, then the closure $\bar{A}$ of the field A in the pseudo-metric space $(2^\Omega, d_\mu)$ is a σ-field.

Pf.