Measure Extension Theorem

$(\Omega, \mathcal{A}, \mu)$ premeasure space.

$\mu^* : 2^\Omega \to [0,\infty]$ Carathéodory outer measure

$$\mu^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

Theorem: (Fréchet, Carathéodory, Hopf, Kolmogorov, ... 1920s)
There is a σ-field $\mathcal{M} \supseteq \mathcal{A}$ s.t. $\mu^*|_{\mathcal{M}}$ is a measure.

$$\therefore\ \sigma(\mathcal{A}) \subseteq \mathcal{M}$$

Standard approach: $\mathcal{M} = \{ E \subseteq \Omega : \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c),\ \forall T \subseteq \Omega \}$

- Show it is a σ-field, containing $\mathcal{A}$
- Show μ^* is countably additive on it

Requires new tool:
Monotone Class Theorem
or
Dynkin's π-λ Theorem

Advantage: works for all premeasures.

Disadvantage: finicky, technical, unmotivated: too clever by half.

Driver's Approach

restrict to finite premeasures.

$(\Omega, \mathcal{A}, \mu)$ $\qquad \mu : \mathcal{A} \to [0, \mu(\Omega)]$

$\qquad\qquad\qquad\qquad \cap$

Want extension: $\bar{\mu} : \bar{\mathcal{A}}$

1. Make 2^Ω into a topological space.
2. Define $\bar{\mathcal{A}}$ to be the closure of $\mathcal{A}$.
3. Prove $\mu : \mathcal{A} \to [0, \infty)$ is sufficiently continuous, $\therefore$ extends to closure.
4. Use topological tools to show $\bar{\mathcal{A}}$ is a σ-field, and $\bar{\mu}$ is a measure.

It will turn out that $\bar{\mu} = \mu^*$ on $\bar{\mathcal{A}}$

Pseudo - Metric Spaces

$d : X \times X \to [0, \infty)$

1. $d(x,y) = 0 \Longleftarrow x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$

Eg. in $\mathbb{R}^2$, $d\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = |x_1 - y_1|$

- A sequence $(x_n)_{n=1}^{\infty}$ in X has a limit x if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \quad d(x_n, x) < \varepsilon .$$

- Given $V \subseteq X$, the closure $\overline{V}$ is the set of limits of sequences in V.
- A set V is closed if $\overline{V} = V$.
- A function $f : V \to \mathbb{R}$ is Lipschitz if $\exists K \in (0, \infty)$ s.t. $|f(x) - f(y)| \leq K d(x,y)$.

Prop: If f is Lipschitz on a nonempty $V \subseteq X$, then there is a unique Lipschitz extension $\bar{f} : \overline{V} \to \mathbb{R}$ (with the same Lipschitz constant K).

$\bar{f}|_V = f$.

The Outer Pseudo-Metric

$(\Omega, \mathcal{A}, \mu)$ finite premeasure space.

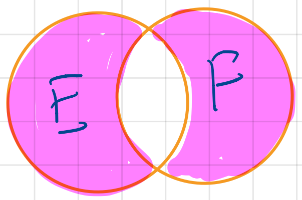
$\mu^* : 2^\Omega \to [0, \mu(\Omega)]$ Carathéodory outer measure

$$\mu^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

Def: $d_\mu : 2^\Omega \times 2^\Omega \to [0, \mu(\Omega)]$

$d_\mu(E, F) = \mu^*(E \Delta F)$

$$E \Delta F = (E \setminus F) \cup (F \setminus E)$$



Prop: d_μ is a pseudo-metric on 2^Ω. ✓

Pf: 0. Takes values in $[0, \mu(\Omega)]$

1. $d_\mu(E, E) = \mu^*(E \Delta E) = \mu^*(\emptyset) = 0$
2. $E \Delta F = F \Delta E$
3. Triangle inequality – HW.

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Key Properties of the Outer Pseudo-Metric

1. $\forall A, B \in 2^{\Omega}$ $\quad d_\mu(A,B) = d_\mu(A^c, B^c)$.

2. $\forall \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in 2^{\Omega}$

(a) $d_\mu\left(\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$

(b) $d_\mu\left(\bigcap_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$.

Pf. 1. $A^c \triangle B^c = (A^c \setminus B^c) \cup (B^c \setminus A^c) = (\underbrace{A^c \cap (B^c)^c}_{B}) \cup (\underbrace{B^c \cap (A^c)^c}_{A}) = (B \setminus A) \cup (A \setminus B) = B \triangle A$

$\therefore d_\mu(A^c, B^c) = \mu^*(A^c \triangle B^c) = \mu^*(B \triangle A) = d_\mu(B, A) = d_\mu(A, B)$.

2. (a) $\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle B_n)$ (HW)

$\therefore d_\mu\left(\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n\right) = \mu^*(\;''\;) \leq \mu^*(\quad) \leq \sum_{n=1}^{\infty} \mu^*(A_n \triangle B_n)$

$= \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$

Lemma: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, and $A_n \in \mathcal{A}$ with $A_n \uparrow A$, then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \to 0$.

Pf. Let $D_n = A_n \setminus A_{n-1}$. Then $A = \bigsqcup_{n=1}^{\infty} D_n$, and by definition of μ^*

$$\mu^*(A) \le \sum_{n=1}^{\infty} \mu(D_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(D_n) = \lim_{N\to\infty} \mu\left(\bigsqcup_{n=1}^{N} D_n\right) = \lim_{N\to\infty} \mu(A_N) \le \mu^*(A)$$

$A_N \in \mathcal{A} \quad \therefore \mu(A_N) = \mu^*(A_N) \le \mu^*(A)$

$$\therefore \lim_{N\to\infty} \mu(A_N) = \mu^*(A).$$

Fix n. Note that $A_N \setminus A_n \uparrow A \setminus A_n$. So repeating $\uparrow$

$$\therefore \mu^*(A \setminus A_n) = \lim_{N\to\infty} \mu(A_N \setminus A_n) = \lim_{N\to\infty} (\mu(A_N) - \mu(A_n))$$

$$= \mu^*(A) - \mu(A_n) \xrightarrow[n\to\infty]{} 0$$

$d_\mu(A, A_n) = \mu^*(A \triangle A_n) = \mu^*(A \setminus A_n)$

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Cor: If $A_n \in \mathcal{A}$ and $A_n \uparrow A$, then $A \in \overline{\mathcal{A}}$.

Theorem: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then the closure $\bar{\mathcal{A}}$ of the field $\mathcal{A}$ in the pseudo-metric space $(2^\Omega, d_\mu)$ is a σ-field.

Pf. 1. $\phi \in \mathcal{A} \subseteq \bar{\mathcal{A}}$ ✓

2. Let $B \in \bar{\mathcal{A}}$. $\therefore \exists A_n \in \mathcal{A}$ s.t. $A_n \xrightarrow{d_\mu} B$. $\therefore$

$$d_\mu(A_n^c, B^c) = d_\mu(A_n, B) \longrightarrow 0. \quad \therefore \underset{\mathcal{A}}{\underset{\cap}{A_n^c}} \longrightarrow B^c \quad \therefore \underset{\bar{\mathcal{A}}}{\underset{\cap}{B^c}}.$$

3. Let $(B_n)_{n=1}^\infty \in \bar{\mathcal{A}}$. Fix $\varepsilon > 0$. For each n, find some $A_n \in \mathcal{A}$ s.t. $d_\mu(A_n, B_n) < \varepsilon/2^n$

$$\therefore d_\mu\Big(\underbrace{\bigcup_{n=1}^\infty A_n}_{\bar{\mathcal{A}}_{1/\varepsilon}}, \bigcup_{n=1}^\infty B_n\Big) < \sum_{n=1}^\infty d_\mu(A_n, B_n) \leq \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon.$$

$\therefore$ For each m find $\bar{A}_m \in \bar{\mathcal{A}}$ s.t.

$$d_\mu\Big(\bar{A}_m, \bigcup_{n=1}^\infty B_n\Big) \leq \frac{1}{m} \quad \therefore \underset{\bar{\mathcal{A}}}{\underset{\cap}{\bar{A}_m}} \xrightarrow{d_\mu} \bigcup_{n=1}^\infty B_n \quad \therefore \underset{\bar{\mathcal{A}}}{\underset{\cap}{}} \quad ///$$