Measure Extension Theorem

$(\Omega, \mathcal{A}, \mu)$ premeasure space.

Want to extend μ to a measure $\bar{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$.

I.e. Find a σ-field $\mathcal{F} \supseteq \sigma(\mathcal{A})$ and a measure $\bar{\mu}$ on $\mathcal{F}$ s.t.

$$\bar{\mu}\big|_{\mathcal{A}} = \mu .$$

Theorem: (Fréchet, 1924)

Every premeasure extends to a measure.
The extension is unique if μ, and $\therefore$ $\bar{\mu}$, is σ-finite.

(Non-) Uniqueness of Extensions.

Eg. $(\Omega, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}_{(1,\infty)})$

Carathéodory's Extension

Let Ω be a set and $\mathcal{E} \subseteq 2^{\Omega}$ s.t. $\emptyset, \Omega \in \mathcal{E}$.

Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$.

Define $\rho^* : 2^{\Omega} \to [0, \infty]$ as follows:

$\rho^*(A) =$

Theorem: If $\mathcal{E}$ is a field and ρ is a premeasure, then ρ^* is a measure.

Proposition: Fix $\rho : \mathcal{E} \subseteq 2^{\Omega} \to [0,\infty]$. $(\phi, \Omega \in \mathcal{E}, \rho(\phi) = 0)$

1. $\rho^*(\phi) = 0$
2. ρ^* is monotone: $A \subseteq B \Rightarrow \rho^*(A) \leq \rho^*(B)$.
3. ρ^* is countably subadditive: $\rho^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \rho^*(A_n)$.

Pf. $\rho^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{E},\ A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$

Def: Let Ω be a nonempty set. A function
$$\nu : 2^{\Omega} \to [0, \infty]$$
is an **outer measure** if:

1. $\nu(\phi) = 0$
2. ν is monotone: $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.
3. ν is countably subadditive: $\nu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \nu(A_n)$.

Thus, Carathéodory's extension ρ^* of a set function $\rho : 2^{\Omega} \to [0, \infty]$ (relative to $2^{\Omega} \supseteq \mathcal{E} \ni \{\phi, \Omega\}$) is an outer measure.

We can use it (in theory) to distinguish finitely-additive measures from premeasures.

Lemma: If $(\Omega, \mathcal{A}, \mu)$ is a premeasure space, and $(\Omega, \sigma(\mathcal{A}), \nu)$ is a measure space extending it, then

$$\nu \leq \mu^* \text{ on } \sigma(\mathcal{A}).$$

Pf.

Prop: If $(\Omega, \mathcal{A}, \chi)$ is a finitely-additive measure space, the $\chi^* \leq \chi$ on $\mathcal{A}$, and $\chi^* = \chi$ on $\mathcal{A}$ iff χ is a premeasure.

Pf. • $A \subseteq A \cup \emptyset \cup \emptyset \cup \cdots$

• If χ is a premeasure, let $A \in \mathcal{A}$, $A \subseteq \bigcup_n A_n$. Then $A = \bigcup_n (A_n \cap A)$

- If $\chi^* = \chi$ on $\mathcal{A}$, then χ is a premeasure.

↳ Let $A_n \in \mathcal{A}$ s.t. $A = \bigsqcup_{n=1}^{\infty} A_n$. Then

$$\chi(A) = \chi^*(A)$$