Measure Extension Theorem

$(\Omega, \mathcal{A}, \mu)$ premeasure space.

Want to extend μ to a measure $\bar{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$.

I.e. Find a σ-field $\mathcal{F} \supseteq \sigma(\mathcal{A})$ and a measure $\bar{\mu}$ on $\mathcal{F}$ s.t.

$$\bar{\mu}\big|_{\mathcal{A}} = \mu .$$

Theorem: (Fréchet, 1924)

Every premeasure extends to a measure.

The extension is unique if μ, and $\therefore$ $\bar{\mu}$, is σ-finite.

$$\Omega = \bigcup_{j \geq 1}^{\infty} A_j \quad \text{s.t. } \mu(A_j) < \infty$$

$\in \mathcal{A}$

(Non-) Uniqueness of Extensions.

Eg. $(\Omega, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}_{(1}, \infty)$

$$\mu(A) = \begin{cases} \infty & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}.$$

Extension $(\Omega, \mathcal{B}(\mathbb{R}), \bar{\mu})$

Another extension

$$\tilde{\mu}(A) = \# A$$

Carathéodory's Extension

Let Ω be a set and $\mathcal{E} \subseteq 2^{\Omega}$ s.t. $\emptyset, \Omega \in \mathcal{E}$.

Let $\rho : \mathcal{E} \to [0,\infty]$ s.t. $\rho(\emptyset) = 0$.

Define $\rho^* : 2^{\Omega} \to [0,\infty]$ as follows:

$$\rho^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{E},\ A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$$

$$A \not\supseteq \bigsqcup_{j=1}^{\infty} E_j \qquad \rho^*(A) = \sum_{j=1}^{\infty} \rho(E_j)$$

$$\bigsqcup_{j=1}^{\infty} E_j \cap \mathcal{E}$$

$$A \subseteq \bigsqcup_{j=1}^{\infty} E_j$$

Theorem: If $\mathcal{E}$ is a field and ρ is a premeasure, then $\rho^*\big|_{\sigma(\mathcal{E})}$ is a measure.

Proposition: Fix $\rho : \mathcal{E} \subseteq 2^{\Omega} \to [0,\infty]$. $(\phi, \Omega \in \mathcal{E}, \rho(\phi) = 0)$

1. $\rho^*(\phi) = 0$ ✓
2. ρ^* is monotone: $A \subseteq B \Rightarrow \rho^*(A) \leq \rho^*(B)$. ✓
3. ρ^* is countably subadditive: $\rho^*\left(\bigcup_{n\geq 1}^{\infty} A_n\right) \leq \sum_{n\geq 1}^{\infty} \rho^*(A_n)$. ✓

Pf. $\rho^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{E},\ A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$

2. $A \subseteq B \subseteq \bigcup_{j=1}^{\infty} E_j$

3. Fix $\{A_n\}$, $\varepsilon > 0$. By def^n of "inf", $\exists \{E_j^n\}$ in $\mathcal{E}$ s.t. $A_n \subseteq \bigcup_j E_j^n$ s.t.

$$\sum_{j=1}^{\infty} \rho(E_j^n) \leq \rho^*(A_n) + \frac{\varepsilon}{2^n}$$

$$\bigcup_n A_n \subseteq \bigcup_{n,j=1}^{\infty} E_j^n$$

Countable

$$\rho^*\left(\bigcup_n A_n\right) \leq \sum_{n\geq 1}^{\infty} \underbrace{\sum_{j=1}^{\infty} \rho(E_j^n)}_{\leq \rho^*(A_n) + \frac{\varepsilon}{2^n}} = \sum_{n\geq 1}^{\infty} \rho^*(A_n) \cancel{+\varepsilon}$$

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Def: Let Ω be a nonempty set. A function

$$\nu : 2^{\Omega} \to [0,\infty]$$

is an **outer measure** if:

1. $\nu(\phi) = 0$
2. ν is monotone: $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.
3. ν is countably subadditive: $\nu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \nu(A_n)$.

Thus, Carathéodory's extension ρ^* of a set function $\rho : 2^{\Omega} \to [0,\infty]$ (relative to $2^{\Omega} \supseteq \mathcal{E} \ni \{\phi, \Omega\}$) is an outer measure.

We can use it (in theory) to distinguish finitely-additive measures from premeasures.

Lemma: If $(\Omega, \mathcal{A}, \mu)$ is a premeasure space, and $(\Omega, \sigma(\mathcal{A}), \nu)$ is a measure space extending it, then

$$\nu \leq \mu^* \text{ on } \sigma(\mathcal{A}).$$

Pf. If $B \in \sigma(\mathcal{A})$, and $E_n \in \mathcal{A}$ s.t. $B \subseteq \bigcup_{n=1}^{\infty} E_n$, then

$$\nu(B) \leq \sum_{j=1}^{n} \nu(E_n) \qquad \therefore \nu(B) \leq \inf\{ \cdots \}$$

$$\nu(E_n) = \mu(E_n) \qquad = \mu^*(B).$$

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Prop: If $(\Omega, \mathcal{A}, \chi)$ is a finitely-additive measure space, the $\chi^* \leq \chi$ on $\mathcal{A}$, and $\chi^* = \chi$ on $\mathcal{A}$ iff χ is a premeasure.

Pf. • $A \subseteq A \cup \phi \cup \phi \cup \cdots \quad \therefore \chi^*(A) \leq \chi(A) + \chi(\phi) + \chi(\phi) + \cdots = \chi(A)$ ✓

• If χ is a premeasure, let $A \in \mathcal{A}$, $A \subseteq \bigcup_n A_n$. Then $A = \bigcup_n \underbrace{(A_n \cap A)}_{B_n \in \mathcal{A}} = \bigcup_n B_n$

$\tilde{B}_1 := B_1$, $\tilde{B}_n := B_n \setminus (B_1 \cup \cdots \cup B_{n-1}) \subseteq B_n$

disjoint

$$\therefore A = \bigsqcup_{n=1}^{\infty} \tilde{B}_n$$

$$\therefore \chi(A) = \sum_{n=1}^{\infty} \chi(\tilde{B}_n) \leq \sum_{n=1}^{\infty} \chi(B_n) \leq \sum_{n=1}^{\infty} \chi(A_n)$$

$$\Rightarrow \chi(A) \leq \chi^*(A).$$

- If $\chi^* = \chi$ on $\mathcal{A}$, then χ is a premeasure.

↳ Let $A_n \in \mathcal{A}$ s.t. $A = \bigsqcup_{n=1}^{\infty} A_n$. Then

$$\underline{\chi(A)} = \chi^*(A) \leq \sum_{n=1}^{\infty} \chi(A_n)$$

$$= \lim_{N \to \infty} \underbrace{\sum_{n=1}^{N} \chi(A_n)}_{\chi\left(\bigsqcup_{n=1}^{N} A_n\right)} \leq \chi(A)$$

$$\bigsqcup_{n=1}^{N} A_n \subseteq A \Rightarrow \chi\left(\bigsqcup_{n=1}^{N} A_n\right) \leq \chi(A)$$

$$\Rightarrow \sum_{n=1}^{\infty} \chi(A_n) = \chi(A)$$