Semi-Algebras of Sets

A collection $\mathcal{S} \subseteq 2^{\Omega}$ is a semi-algebra

(1) $\emptyset \in \mathcal{S}$

(2) If $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$

(3) If $A \in \mathcal{S}$ then A^c is a finite disjoint union of elements from $\mathcal{S}$.

The canonical example is

$$\mathcal{S} = \{(a,b] : -\infty \leq a \leq b \leq \infty\}$$

Prop: If $\mathcal{S}$ is a semi-algebra over Ω, then
$A(\mathcal{S}) = \{\text{finite disjoint unions of sets from } \mathcal{S}\}$

Prop: If $\chi : \mathcal{S} \to [0,\infty]$ is additive over disjoint unions, then

$$\chi\left(\bigsqcup_{j=1}^{n} E_j\right) := \sum_{j=1}^{n} \chi(E_j)$$

is a well-defined finitely-additive measure on $A(\mathcal{S})$.

Stieltjes Premeasures?

For $F : \mathbb{R} \to \mathbb{R}$ non-decreasing,

$$\chi_F(a,b] = F(b) - F(a)$$

is additive, and so extends to a finitely-additive measure on $\mathcal{B}_{(]}(\mathbb{R})$.

It is not a premeasure if F fails to be right-continuous, as we saw.

Fortunately, the converse is true.

Theorem: The finitely-additive measure χ_F is a premeasure (i.e. is countably additive) on $\mathcal{B}_{(]}(\mathbb{R})$ iff F is right-continuous on $\mathbb{R}$:

$$\lim_{\delta \downarrow 0} F(a+\delta) = F(a).$$

Prop: Let $\mathcal{S} \subseteq 2^{\Omega}$ be a semi-algebra.
A finitely-additive measure $\chi : \mathcal{A}(\mathcal{S}) \to [0,\infty]$ is a premeasure iff it is countably subadditive on $\mathcal{S}$:

$$E = \bigsqcup_{j=1}^{\infty} E_j \text{ in } \mathcal{S} \Rightarrow \chi(E) \leq \sum_{j=1}^{\infty} \chi(E_j)$$

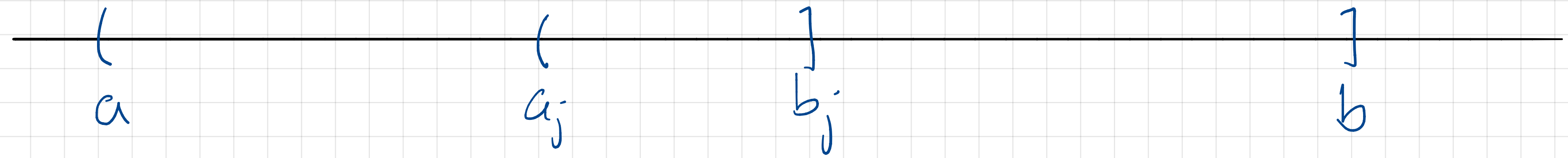
Pf. $(\Rightarrow)$ Premeasures are countably additive.

$(\Leftarrow)$ Finitely-additive measures are always countably superadditive, so it suffices to prove that χ is countably subadditive on $\mathcal{A} = \mathcal{A}(\mathcal{S})$

$$A = \bigsqcup_{n=1}^{\infty} A_n$$

We now show that $\mu_F : \mathcal{A}(\mathcal{I}_{(\,]}) = \mathcal{B}_{(\,]}(\mathbb{R}) \to [0,\infty)$ is a premeasure by showing it is countably subadditive on the semi-algebra $\mathcal{I}_{(\,]} = \{(a,b]\}$

$$(a,b] = \bigsqcup_{j=1}^{\infty} (a_j, b_j]$$



$\therefore$ λ_F is a premeasure on $\mathcal{B}_{(\,]}(\mathbb{R})$.

Notably: $F(x) = x$ $\lambda(a,b] = b-a$

Lebesgue premeasure.

$\lambda(E+\alpha)$