Semi-Algebras of Sets

A collection $\mathcal{S} \subseteq 2^{\Omega}$ is a semi-algebra

(1) $\emptyset \in \mathcal{S}$

(2) If $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$

(3) If $A \in \mathcal{S}$ then A^c is a finite disjoint union of elements from $\mathcal{S}$.

The canonical example is

$$\mathcal{S} = \{(a,b] : -\infty \leq a \leq b \leq \infty\} \qquad \mathcal{A}(\mathcal{S}) = \mathcal{B}_{(]}(\mathbb{R})$$

Prop: If $\mathcal{S}$ is a semi-algebra over Ω, then $\mathcal{A}(\mathcal{S}) = \{\text{finite disjoint unions of sets from } \mathcal{S}\}$

Prop: If $\chi : \mathcal{S} \to [0,\infty]$ is additive over disjoint unions, then

$$\chi\left(\bigsqcup_{j=1}^{n} E_j\right) := \sum_{j=1}^{n} \chi(E_j)$$

is a well-defined finitely-additive measure on $\mathcal{A}(\mathcal{S})$.

Stieltjes Premeasures?

For $F : \mathbb{R} \to \mathbb{R}$ non-decreasing,

$$\chi_F(a,b] = F(b) - F(a)$$

is additive, and so extends to a finitely-additive measure on $\mathcal{B}_{(]}(\mathbb{R})$.

It is not a premeasure if F fails to be right-continuous, as we saw.

Fortunately, the converse is true.

Theorem: The finitely-additive measure χ_F is a premeasure (i.e. is countably additive) on $\mathcal{B}_{(]}(\mathbb{R})$ iff F is right-continuous on $\mathbb{R}$:

$$\lim_{\delta \downarrow 0} F(a+\delta) = F(a).$$

Prop: Let $\mathcal{S} \subseteq 2^{\Omega}$ be a semi-algebra.
A finitely-additive measure $\chi : \mathcal{A}(\mathcal{S}) \to [0,\infty]$ is a premeasure iff it is countably subadditive on $\mathcal{S}$:

$$E = \bigsqcup_{j=1}^{\infty} E_j \text{ in } \mathcal{S} \Rightarrow \chi(E) \leq \sum_{j=1}^{\infty} \chi(E_j)$$

Pf. $(\Rightarrow)$ Premeasures are countably additive. ✓

$(\Leftarrow)$ Finitely-additive measures are always countably superadditive, so it suffices to prove that χ is countably subadditive on $\mathcal{A} = \mathcal{A}(\mathcal{S})$

$$\mathcal{A} \ni A = \bigsqcup_{n=1}^{\infty} A_n \in \mathcal{A}, \quad \bigsqcup_{j=1}^{N} E_j = A, \quad A_n = \bigsqcup_{i=1}^{N_n} E_i^n$$

$$E_j = \bigsqcup_n A_n \cap E_j = \bigsqcup_n \bigsqcup_{i=1}^{N_n} \underbrace{E_i^n \cap E_j}_{\in \mathcal{S}}$$

$$\chi(E_j) \leq \sum_n \sum_i \chi(E_i^n \cap E_j)$$

$$\underbrace{\sum_{j=1}^{N} \chi(E_j)}_{\chi(A)} \leq \sum_{j=1}^{N} \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \chi(E_i^n \cap E_j) = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \underbrace{\sum_{j=1}^{N} \chi(E_i^n \cap E_j)}_{\chi(E_i^n)} = \sum_{n=1}^{\infty} \chi(A_n)$$

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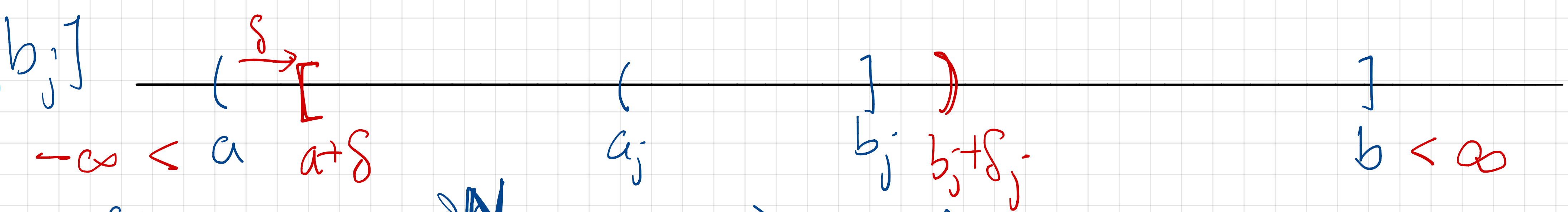
We now show that $\mu_F : \mathcal{A}(\mathcal{E}_{(\,]}) = \mathcal{B}_{(\,]}(\mathbb{R}) \to [0,\infty)$ is a premeasure by showing it is countably subadditive on the semi-algebra $\mathcal{E}_{(\,]} = \{(a,b]\}$

$$(a,b] = \bigsqcup_{j=1}^{\infty} (a_j, b_j]$$



compact

$$[a+\delta, b] \subset (a,b] = \bigsqcup_{j=1}^{\infty} (a_j, b_j] \subseteq \bigcup_{j=1}^{N} (a_j, b_j+\delta_j) \subseteq \bigcup_{j=1}^{N} (a_j, b_j+\delta_j]$$

$\therefore \exists N < \infty$ s.t.

μ_F is subadditive

$$\mu_F(a+\delta, b] \le \sum_{j=1}^{N} \mu_F(a_j, b_j+\delta_j] \le \sum_{j=1}^{\infty} \mu_F(a_j, b_j+\delta_j]$$

$$\mu_F(a_j, b_j] + \mu_F(b_j, b_j+\delta_j]$$

$$\mu_F(a+\delta, b] \le \sum_{j=1}^{\infty} \mu_F(a_j, b_j] + \underbrace{\sum_{j=1}^{\infty} \overbrace{\mu_F(b_j, b_j+\delta_j]}^{F(b_j+\delta_j)-F(b_j)}}_{< \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.}$$

$$\overset{\|}{F(b) - F(a+\delta)}$$

$$\Big\downarrow \delta \downarrow 0$$

$$\mu_F(a,b] = F(b) - F(a)$$

Fix $\varepsilon > 0$

$\therefore \delta_j > 0$ s.t.

$$F(b_j+\delta_j) - F(b_j) < \frac{\varepsilon}{2^j}.$$

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$\therefore\ \lambda_F$ is a premeasure on $\mathcal{B}_{(]}(\mathbb{R})$.

Notably: $F(x) = x \qquad \lambda(a,b] = b-a$

Lebesgue premeasure.

$$\lambda(E+\alpha) = \sum_j \lambda(a_j+\alpha, b_j+\alpha] = \sum_j \left[(b_j+\cancel{\alpha}) - (a_j+\cancel{\alpha})\right]$$

$$= \sum_j b_j - a_j = \sum_j \lambda(a_j, b_j]$$

$$= \lambda(E).$$

$$E = \bigsqcup_j (a_j, b_j]$$

$$E+\alpha = \bigsqcup_j (a_j+\alpha, b_j+\alpha]$$