Math 280A: Fall 2020
Homework 9

Available	Monday, November 30	Due	Monday, December 7

Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

1. In this exercise, you will show that moment factorization is equivalent to independence for *bounded* random variables. To make notation easier, you'll do it for two random variables; the same proof idea works for arbitrary collections. Let X and Y be Borel random variables satisfying $|X| \le M$ and $|Y| \le M$ a.s. for some $M < \infty$. Assume that

$$\mathbb{E}[X^n Y^m] = \mathbb{E}[X^n]\mathbb{E}[Y^m], \qquad \forall\, n, m \in \mathbb{N}.$$

(a) Show that $\mathbb{E}[p(X)q(Y)] = \mathbb{E}[p(X)]\mathbb{E}[q(Y)]$ for all polynomials p and q.

(b) Let $\mathbb{M}$ denote the set of functions $f \in C_c(\mathbb{R})$ with the property that f is equal to a polynomial on $[-M, M]$. Prove that $\mathbb{M}$ is a multiplicative system, and prove that $\sigma(\mathbb{M}) = \mathcal{B}(\mathbb{R})$.

(c) fix a polynomial q, and let $\mathbb{H}_q$ denote the set of all $f \in \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with the property that $\mathbb{E}[f(X)q(Y)] = \mathbb{E}[f(X)]\mathbb{E}[q(Y)]$. Show that $\mathbb{H}_q$ contains all bounded Borel functions.

The remainder of the proof involves repeating the same steps, this time holding $f \in \mathbb{B}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ fixed and showing that the set of g for which $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ includes all bounded Borel functions, therefore showing that X and Y are independent.

2. Let $\mu_1, \ldots, \mu_d$ be Borel probability measures on $\mathbb{R}$, and let λ denote the Lebesgue measure on $\mathbb{R}$. Suppose that $\mu_j \ll \lambda$ for $j \in [d]$. Show that $\mu = \mu_1 \otimes \cdots \otimes \mu_d$ is absolutely continuous with respect to λ^d (the Lebesgue measure on $\mathbb{R}^d$), and that

$$\frac{d\mu}{d\lambda^n} = \frac{d\mu_1}{d\lambda} \otimes \cdots \otimes \frac{d\mu_d}{d\lambda}.$$

3. (Driver, Exercise 13.1) Let ν_n be probability measures on the cubes $([0,1]^n, \mathcal{B}([0,1]^n))$, satisfying the consistency condition

$$\nu_{n+1}(A \times [0,1]) = \nu_n(A), \qquad \forall\, n \in \mathbb{N},\; A \in \mathcal{B}([0,1]^n).$$

Show that $P(A \times [0,1]^{\mathbb{N}}) := \nu_n(A)$ defines a well-defined, finitely-additive measure P on the algebra $\mathcal{A} = \bigcup_{n\in\mathbb{N}}\{A \times [0,1]^{\mathbb{N}} \colon A \in \mathcal{B}([0,1]^n)\}$.

4. Let $\Omega = (0,1)$, equipped with the Borel σ-field $\mathcal{B} = \mathcal{B}(0,1)$, and the Lebesgue measure λ (which is a probability measure on $(0,1)$). Let $X_n(\omega) = n^{\text{th}}$ binary digit of ω; i.e.

$$X_n(\omega) = \begin{cases} 1 & \text{if } \lfloor 2^n \omega \rfloor \text{ is odd,} \\ 0 & \text{if } \lfloor 2^n \omega \rfloor \text{ is even.} \end{cases}$$

Prove that $\{X_n\}_{n=1}^{\infty}$ is an i.i.d. sequence of Bernoulli random variables on $(\Omega, \mathcal{B}, \lambda)$.

5. Suppose $\{X_n\}_{n=1}^{\infty}$ are independent. Show that $\mathbb{P}(\limsup_{n\to\infty} X_n > 1)$ is either 0 or 1.