Math 280A: Fall 2020
Homework 7

Available	Monday, November 16	Due	Monday, November 23

Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

1. (Driver, Exercise 10.5) Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are C^1 functions, for which all three functions $f'g$, fg',and fg are in $L^1(\mathbb{R}, \mathcal{B}, \lambda)$. Prove the *integration by parts* formula:

$$\int_{\mathbb{R}} f' g \, d\lambda = -\int_{\mathbb{R}} f g' \, d\lambda.$$

[*Hint*: Let $\psi \in \mathbb{C}^1(\mathbb{R})$ be a non-negative function with $\psi(x) = 1$ when $|x| \le 1$, $\psi(x) = 0$ when $|x| \ge 2$, and $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}$. Set $\psi_n(x) = \psi(x/n)$. Use ordinary calculus to prove the above formula applied to the functions $f\psi_n$ and $g\psi_n$, then use the DCT to prove the result in general.]

2. Let X_n be a sequence of random variables, and let $a_n \in \mathbb{R}$ be a convergent sequence with $\lim_{n\to\infty} a_n = a$. If $X_n - a_n \to_{\mathbb{P}} 0$, prove that $X_n \to_{\mathbb{P}} a$.

3. For $X, Y \in L^0(\Omega, \mathcal{F}, \mathbb{P})$, define

$$d(X, Y) := \mathbb{E}[\min\{|X - Y|, 1\}].$$

- **(a)** Prove that d is a metric on L^0.
- **(b)** Let $X_n, X \in L^0$. Show that $X_n \to_{\mathbb{P}} X$ if and only if $d(X_n, X) \to 0$. [*Hint*: break up the integral over $\{|X_n - X| \ge \epsilon\}$ and its complement.]

4. (Driver, Exercise 12.4) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X, Y\colon \Omega \to \mathbb{R}$ be random variables with the property that

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)g(X)]$$

for all bounded measurable functions f, g. Show that $\mathbb{P}(X = Y) = 1$. [*Hint*: Show that if $\mathbb{H}$ is the collection of all bounded measurable functions $h\colon \mathbb{R}^2 \to \mathbb{R}$ for which $\mathbb{E}[h(X, Y)] = \mathbb{E}[h(X, X)]$, then in fact $\mathbb{H}$ contains *all* bounded measurable functions, including $h(x, y) = \mathbb{1}_{\{x=y\}}$.]

5. (Driver, Exercise 12.5) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{A} \subset \mathcal{F}$ be an algebra such that $\sigma(\mathcal{A}) = \mathcal{F}$. An $\mathcal{A}$**-simple function** is a simple function φ for which $\varphi^{-1}\{t\} \in \mathcal{A}$ for every $t \in \mathbb{R}$. Prove that, for any bounded random variable X, and any $\epsilon > 0$, there is an $\mathcal{A}$-simple function φ with $\mathbb{E}[|X - \varphi|] < \epsilon$.