Math 280A: Fall 2020
Homework 6

Available	Monday, November 9	Due	Monday, November 16

Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\varrho_1, \varrho_2 \colon \Omega \to [0,\infty)$ be in $L^1(\Omega, \mathcal{F}, \mu)$. Suppose that, for all $A \in \mathcal{F}$,

$$\int_A \varrho_1 \, d\mu = \int_A \varrho_2 \, d\mu.$$

Prove that $\varrho_1 = \varrho_2$ a.s.$[\mu]$. [*Hint*: Use Problem **4(c)** from HW5, with the function $f = \varrho_1 - \varrho_2$.]

2. (Dirver, Exercise 10.7) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(S, \mathcal{B})$ be a measurable space. Let $X \colon \Omega \to S$ be a measurable function, and let $\nu = X_*\mu$, i.e. ν is the measure on $(S, \mathcal{B})$ defined by $\nu(A) = \mu(X^{-1}(A))$.

(a) Show that

$$\int_S g \, d\nu = \int_\Omega (g \circ X) \, d\mu \qquad (*)$$

for all measurable functions $g \colon S \to [0, \infty]$. [*Hint*: prove it first for simple functions g, and then for non-negative measurable functions g with an appropriate convergence theorem.]

(b) Show that a measurable function $g \colon S \to \mathbb{R}$ is in $L^1(S, \mathcal{B}, \nu)$ if and only if $g \circ X \in L^1(\Omega, \mathcal{F}, \mu)$, and that in this case, Equation $(*)$ still holds for g.

3. Let X be a standard normal random variable: $X \stackrel{d}{=} \mathcal{N}(0,1)$. Let $f \in C^1(\mathbb{R})$, with the property that $Xf(X)$, $f'(X)$, and $f(X)$ are all integrable random variables. Prove that

$$\mathbb{E}[Xf(X)] = \mathbb{E}[f'(X)].$$

[*Fun fact*: this property actually characterizes the standard normal distribution $\mathcal{N}(0,1)$.]

4. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a random variable. Prove that, for any $\epsilon > 0$, there is a *simple* random variable $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[|X - Y|] < \epsilon$.

5. Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a non-negative random variable. Show that

$$\mathbb{P}(X > 0) \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$

[*Hint*: Apply the Cauchy–Schwarz Inequality to $X\mathbb{1}_{X>0}$.]