Math 280A: Fall 2020
Homework 5

Available	Monday, November 2	Due	Monday, November 9

Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

1. Let X_n be a sequence of random variables, for which $\sum_{n=1}^{\infty} \mathbb{E}[|X_n|] < \infty$. Prove that

$$\lim_{n\to\infty} X_n = 0 \text{ a.s.}$$

[*Hint*: Use Markov's inequality to estimate $\sum_{n=1}^{\infty} \mathbb{P}\{|X_n| \geq \epsilon\}$ for any $\epsilon > 0$.]

2. (Dirver, Exercise 10.1) Suppose that $(\Omega, \mathcal{F})$ is a measurable space, and $\mu_n \colon \mathcal{F} \to [0, \infty]$ are measures for $n \in \mathbb{N}$. Suppose further that, for each $A \in \mathcal{F}$, the sequence $\mu_n(A)$ is non-decreasing. Prove that

$$\mu(A) := \lim_{n\to\infty} \mu_n(A)$$

defines a measure on $(\Omega, \mathcal{F})$.

3. Prove the following generalization of the MCTheorem, beyond positive functions: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f_n \in L^1(\Omega, \mathcal{F}, \mu)$, not necessarily non-negative, such that $f_n \leq f_{n+1}$ μ-a.s. for each n, and suppose $f_n \to f$ a.s. as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

[*Hint*: Consider $f_n - f_1$.]

4. (Driver, Exercise 10.6) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\varrho \colon \Omega \to [0, \infty]$ be a Borel-measurable function. Define a function $\nu \colon \mathcal{F} \to [0, \infty]$ by

$$\nu(A) = \int_A \varrho \, d\mu.$$

(a) Show that ν is a measure on $\mathcal{F}$.

(b) If $f \colon \Omega \to [0, \infty]$ is Borel measurable, show that

$$\int_\Omega f \, d\nu = \int_\Omega f \varrho \, d\mu. \qquad (*)$$

[*Hint*: prove it first for simple functions f, and then for non-negative measurable functions f with an appropriate convergence theorem.]

(c) For any $f \colon \Omega \to \mathbb{R}$, show that $f \in L^1(\nu)$ if and only if $|f|\varrho \in L^1(\mu)$, and in this case $(*)$ still holds for f.

5. (Driver, Exercise 10.29) Let λ denote Lebesgue measure on $\mathbb{R}$, and let $f \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. For each $x \in \mathbb{R}$, define

$$F(x) = \int_{-\infty}^{x} f(t)\,\lambda(dt) = \int_{(-\infty,x]} f\,d\lambda.$$

Show that F is a continuous function. Is this still true if you replace λ by any Radon measure on $\mathbb{R}$? Prove or disprove your answer.