Math 280A: Fall 2020
Homework 4

Available	Monday, October 26	Due	Monday, November 2

Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

1. Let $(\Omega, \mathcal{F})$ be a measurable space, and let $Y_1, \ldots, Y_n \colon \Omega \to \mathbb{R}$ be $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable functions. Let $A_1, \ldots, A_n$ be disjoint events in $\mathcal{F}$ that partition Ω: $\Omega = \bigsqcup_{j=1}^n A_j$. Define a function $X \colon \Omega \to \mathbb{R}$ by $X(\omega) = Y_j(\omega)$ if $\omega \in A_j$. Prove that X is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable.

2. Let Ω be a set, and let $X_1, \ldots, X_d \colon \Omega \to \mathbb{R}$ be functions. Recall that the σ-field generated by $X_1, \ldots, X_d$, denoted $\mathcal{F} = \sigma(X_1, \ldots, X_d)$, is the smallest σ-field over Ω with respect to which $X_1, \ldots, X_d$ are Borel measurable (i.e. X_j is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable for $1 \le j \le d$). Let $\mathbf{X} = (X_1, \ldots, X_d)$ be the $\mathbb{R}^d$-valued function whose components are the X_j. Prove that

$$\sigma(X_1, \ldots, X_d) = \mathbf{X}^*\mathcal{B}(\mathbb{R}^d) = \{\mathbf{X}^{-1}(B) \colon B \in \mathcal{B}(\mathbb{R}^d)\}.$$

3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\beta_1, \ldots, \beta_m \in \mathbb{R}$ and let $B_1, \ldots, B_m \in \mathcal{F}$, *not necessarily disjoint*. Show that the function $g = \sum_{j=1}^m \beta_j \mathbb{1}_{B_j}$ is a simple measurable function, and that

$$\int g \, d\mu = \sum_{j=1}^m \beta_j \, \mu(B_j).$$

4. (Exercise 5.6 in Driver) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $d_\mu(A, B) = \mathbb{P}(A \triangle B)$ for all $A, B \in \mathcal{F}$.

(a) Show that $d_\mu(A, B) = \mathbb{E}\left[|\mathbb{1}_A - \mathbb{1}_B|\right]$.

(b) Use part **(a)** to give another proof that d_μ satisfies the triangle inequality.

5. (Exercise 5.12 in Driver) Let $F, G \colon [0,1] \to \mathbb{R}$ be two non-decreasing functions with $F(0) = G(0)$ and $F(1) = G(1)$, and suppose that $\{x \in [0,1] \colon F(x) \ne G(x)\}$ is countable. Prove that, as Riemann–Stieltjes integrals,

$$\int_0^1 f \, d\mu_F = \int_0^1 f \, d\mu_G, \qquad \text{for all continuous functions } f \colon [0,1] \to \mathbb{R}.$$

On the other hand, under the above circumstances, show that the finitely-additive measures μ_F and μ_G are equal iff $F = G$. (They are only countably-additive measures if F and G are right-continuous.)