Math 280A: Fall 2020
Homework 3

Available	Monday, October 19	Due	Monday, October 26

Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

1. (Exercise 2.13 in Driver) Let (X,d) be a pseudo-metric space. Let $V \subseteq X$ be a nonempty subset, and suppose $f\colon V \to \mathbb{R}$ is a Lipschitz function: there is a constant $K \in (0,\infty)$ such that

$$|f(x) - f(y)| \le K\, d(x,y), \qquad \forall\, x,y \in V.$$

Prove that there is a unique extension $\bar{f}$ of f to $\overline{V}$ that is also Lipschitz (with the same constant K).

2. Let λ denote the Lebesgue pre-measure on the "Borel field" $\mathcal{B}_{(]}(\mathbb{R})$, and let λ^* denote its Carathéodory outer measure on $2^{\mathbb{R}}$. Prove that λ^* is translation-invariant: $\lambda^*(V+\tau) = \lambda^*(V)$ for all $\tau \in \mathbb{R}$. Conclude that λ^* is *not* a measure: it cannot be countably additive over disjoint subsets of $2^{\mathbb{R}}$.

3. Fix a real number $\gamma \neq 0$.

(a) If $B \in \mathcal{B}(\mathbb{R})$ is a Borel set, show that $\gamma \cdot B = \{\gamma x \colon x \in B\}$ is a Borel set. [*Hint*: First prove this for half-open intervals B, then use [Driver, Lemma 9.3].]

(b) Let μ be a Borel measure on $\mathbb{R}$. Define $\nu\colon \mathcal{B}(\mathbb{R}) \to [0,\infty]$ by

$$\nu(B) = \frac{1}{|\gamma|}\mu(\gamma \cdot B).$$

Prove that ν is a Borel measure.

(c) If μ is translation-invariant, prove that ν is also translation invariant.

4. On $\mathbb{R}^2$, consider the set of all "half-open rectangles":

$$\mathcal{R} = \{(a_1,b_1] \times (a_2,b_2] \colon -\infty \le a_j \le b_j \le \infty,\ j = 1,2\}.$$

(a) Show that $\mathcal{R}$ is a semi-algebra.

(b) Show that $\sigma(\mathcal{R}) = \mathcal{B}(\mathbb{R}^2)$, the full Borel σ-field on the plane, generated by all open balls. [*Hint*: show that every open ball in $\mathbb{R}^2$ is in $\mathcal{R}_\sigma$.]

5. (Exercise 9.1 in Driver) Let $(\Omega,\mathcal{F})$ and $(S,\mathcal{B})$ be measurable spaces, and let $f\colon \Omega \to S$ be a function. Show that

$$f^*\mathcal{B} := \{f^{-1}(B) \colon B \in \mathcal{B}\} \subseteq 2^\Omega$$
$$f_*\mathcal{F} := \{E \subseteq S \colon f^{-1}(E) \in \mathcal{F}\} \subseteq 2^S$$

are σ-fields.