Math 280A: Fall 2020

Homework 2

| Available | Monday, October 12 | Due | Monday, October 19 |
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Turn in the homework by 9:00pm on Gradescope. Late homework will not be accepted.

**1.** Let $\Omega = \{1,2,3,4\}$, and set $\mathcal{S} = \{\varnothing, \{1\}, \{2\}, \{3,4\}, \Omega\}$.

**(a)** Prove that $\mathcal{S}$ is a semi-algebra.

**(b)** Define $\chi\colon \mathcal{S} \to \mathbb{R}$ as follows:

$$\chi(\varnothing) = 0, \chi(\{1\}) = \chi(\{2\}) = \chi(\{3,4\}) = 1, \chi(\Omega) = 4.$$

Show that $\chi$ is "pairwise" additive: $\chi(A \sqcup B) = \chi(A) + \chi(B)$ whenever $A$, $B$, and $A \sqcup B$ are all in $\mathcal{S}$. Show also that $\chi$ is *not* additive over all finite disjoint unions.

(This shows it is important, when dealing with semi-algebras and other classes not closed under finite union, to spell out the full statement of "finite additivity" in all proofs.)

**2.** (Exercise 4.12 in Driver) Let $\Omega_1$ and $\Omega_2$ be sets, and let $\mathcal{A}_1 \subseteq 2^{\Omega_1}$ and $\mathcal{A}_2 \subseteq 2^{\Omega_2}$ be semi-algebras. Show that

$$\mathcal{S} = \mathcal{A}_1 \times \mathcal{A}_2 = \{A_1 \times A_2 \colon A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\} \subset 2^{\Omega_1 \times \Omega_2}$$

is a semi-algebra.

**3.** (Exercise 4.3 in Driver) Let $A_n, B_n \subseteq \Omega$ for $n \in \mathbb{N}$. Show that

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \setminus B_n).$$

Use this to show that

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle B_n).$$

**4.** (Exercise 4.4 in Driver) Let $A, B, C \subseteq \Omega$. Recall that the **symmetric difference** of sets is $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$.

**(a)** Show that $A \cap C^c \subseteq (A \cap B^c) \cup (B \cap C^c)$.

**(b)** Use part **(a)** to show that

$$A \triangle C \subseteq (A \triangle B) \cup (B \triangle C).$$

**(c)** Now, let $\upsilon\colon 2^{\Omega} \to [0,\infty)$ be an outer measure. Show that the function $d\colon 2^{\Omega} \times 2^{\Omega} \to [0,\infty)$ defined by $d(A,B) = \upsilon(A \triangle B)$ satisfies the **triangle inequality**:

$$d(A,C) \leq d(A,B) + d(B,C).$$

**5.** Let $\mathcal{A}$ be a field over $\Omega$, and let $\mathbb{P}$ be a probability measure on $(\Omega, \sigma(\mathcal{A}))$. Let $B \in \sigma(\mathcal{A})$. Prove that for any $\epsilon > 0$, there is a set $A \in \mathcal{A}$ such that $\mathbb{P}(A \triangle B) < \epsilon$. I.e. $\mathcal{A}$ is "dense" in $\sigma(\mathcal{A})$. [*Hint*: show that the collection of all sets $B$ satisfying this property is a $\sigma$-field.]