MATH 280A: Probability Theory I

Tuesday, Dec 8, 2020

* **Video Lectures:** 18.1, 18.2, 19.1, 19.2 posted on YouTube

* **Quizzes:** Regrade requests now active for all 5 quizzes. Will remain so until Dec 20 @ 8pm PST.

* **Final Exam:** Take-Home. Available: Sunday, Dec 13 @ 2:30pm PST.
  Due: Friday, Dec 18 @ 2:30pm PST.

* **Exit Survey:** Google form posted and emailed to you (shortly). Your feedback is appreciated!

$\{X_n\}_{n\geq 1}^{\infty}$ indep. rv's.

$\{\limsup_{n\to\infty} X_n > 1\} = \{\forall N\in\mathbb{N}\ \exists n\geq N \text{ s.t. } X_n > 1\}$

Fix $N\in\mathbb{N}$, $\tilde{X}_n^N = \begin{cases} 0 & \text{if } n < N \\ X_n & \text{if } n\geq N \end{cases}$

$= X_n \mathbb{1}_{n\geq N}$.

$= \{\limsup_{n\to\infty} \tilde{X}_n^N > 1\} \in \sigma(\tilde{X}_1^N, \tilde{X}_2^N, \dots, \tilde{X}_N^N, \tilde{X}_{N+1}^N, \dots)$

$= \sigma(0, 0, \dots, 0, X_N, X_{N+1}, \dots)$

$= \sigma(X_N, X_{N+1}, \dots)$

$\therefore \{\limsup_{n\to\infty} X_n > 1\} \in \bigcap_{N=1}^{\infty} \sigma(X_N, X_{N+1}, \dots)$

$= \mathcal{T}$

$\{\limsup_{n\to\infty} X_n \geq 1\}$

$\{\forall \varepsilon > 0, \forall N\in\mathbb{N}\ \exists n\geq N \text{ s.t. } X_n > 1-\varepsilon\}$

$= \{\limsup_{n\to\infty} \tilde{X}_n^N \geq 1\}$

$\limsup_{n\to\infty} X_n = \lim_{n\to\infty} \sup_{k\geq n} X_k$

$\forall n \quad X_1, \dots, X_n$ discrete.

indep. iff $\quad \mathbb{P}(X_1 = k_1, X_2 = k_2, \dots, X_n = k_n) \qquad k_1, \dots, k_n \in \{0, 1\}$

$= \mathbb{P}(X_1 = k_1) \cdots \mathbb{P}(X_n = k_n) = \frac{1}{2} \cdot \dots \cdot \frac{1}{2}$

$\{0.k_1 * * * \dots\}$ $\quad$ $\{X_2 = k_2\} = \{0.*k_2 * * * \dots\}$

$k_1 = 0 \quad k_1 = 1$ $\quad$ $k_2 = 0 \quad k_2 = 0$ $\quad$ $k_2 = 1 \quad k_2 = 1$

$\{X_1 = k_1, \dots, X_n = k_n\} = \{0.k_1 k_2 \dots k_n * * * * * \dots\}$

$\lambda(\qquad\qquad) = \frac{1}{2^n}$

$$d\mu_j = f_j \, d\lambda \qquad 1 \le j \le d$$

$$d\mu_1 \otimes \cdots \otimes \mu_d = (f_1 \otimes \cdots \otimes f_d)\, d\lambda^d$$

$$f_1(x_1)\, f_2(x_2) \cdots f_d(x_d)$$

$$\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$$
$$= \sigma\big(B_1 \times B_2 : B_1, B_2 \in \mathcal{B}(\mathbb{R})\big)$$
$$= \sigma\big((-\infty, s] \times (-\infty, t] : s, t \in \mathbb{R}\big)$$

$$\underset{\in \mathcal{B}(\mathbb{R}^d)}{\mu_1 \otimes \cdots \otimes \mu_d (B)} = \int_B f \, d\lambda^d$$

$f \in L^1$ $\quad f \ge 0$

$$\int_{\mathbb{R}^d} f \, d\lambda^d = 1.$$

$\forall B \in \mathcal{B}(\mathbb{R}^d)$

suffices to show for

$B = B_1 \times \cdots \times B_d$, $B_j \in \mathcal{B}(\mathbb{R})$

$$\mu_1(B_1) \cdots \mu_d(B_d) = \int_{B_1 \times \cdots \times B_d} f \, d\lambda^d$$

Quiz 5 Prob. 3 : X_n, X $\mathbb{R}$-valued

$X_n \to X$ a.s.

☑ $X_n \to_{\mathbb{P}} X$ ✓

☒ $X_n \to X$ in L^1 $\quad X_n = X \notin L^1$

☒ A subseq. $X_{n_k} \to X$ in L^1

☑ If $\exists$ const. $M < \infty$, s.t. $|X_n| \le M$ a.s. $\forall$ large n, ✓

$X_n \to X$ in L^p $\forall p \ge 1$. $\quad |X_n|^p \le M^p < \infty \quad \forall p \ge 1$.

$\therefore \uparrow_{\text{by DCT}}$

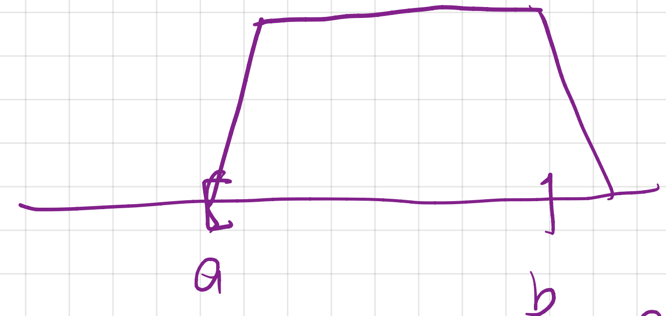
Quiz 5 Problem 4:

X_n, X $\mathbb{R}$-valued, $X_n \to_p X$.

- ✓ Every subseq. of X_n conv. to X in prob.
- ✗ Every subseq. of X_n conv. to X in L^1 E.g. $X_n = X \notin L^1$
- ✓ $\exists$ subseq. $X_{n_k} \to X$ a.s.
- ✗ $\exists$ subseq. $X_{n_k} \to X$ in L^1.

Quiz 5 Problem 6: Closed under Borel Conv.?

$\cancel{\boxed{\ }}$ $C_b(\mathbb{R})$



$\in C_b(\mathbb{R}) \longrightarrow$ $[a, b] \notin C(\mathbb{R})$

$\boxed{\checkmark}$ $B(\mathbb{R}, \mathcal{B}(\mathbb{R})) \ni f_n$, $|f_n| \le M$, $f_n \to f \Rightarrow |f| \le M$, f is $\mathcal{B}/\mathcal{B}$-meas.

$\boxed{\checkmark}$ $\{ f \ge 0,\ f \in B([0,1], \mathcal{B}([0,1]))$ s.t. $\int_0^1 f\,d\lambda = 1 \}$

$\cancel{\boxed{\ }}$ $\{ f \ge 0,\ f \in B(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $\int_{\mathbb{R}} f\,d\lambda = 1 \}$. $f_n = \mathbb{1}_{[n, n+1]}$

$\cancel{L^1(\mathbb{R}, \lambda)}$ $f_n \to 0$

$|f_n| \le 1$.

$|f_n| \le M$ $\quad \int f_n\,d\lambda = 1$.

$f_n \quad f_n \to f$ a.s. $\overset{?}{\Rightarrow} \int_0^1 f\,d\lambda = \int_0^1 \lim_{n\to\infty} f_n\,d\lambda \overset{DCT}{=} \lim_{n\to\infty} \int_0^1 f_n\,d\lambda$

$= 1$.