MATH 280A: Probability Theory I

Tuesday, Dec 1, 2020

* **Video Lectures:** 15.1, 15.2, 16.1, 16.2, 17.1, 17.2 posted on YouTube

* **Quiz 5:** This Thursday @ 1-1:50pm or 7-7:50pm (Google Sheet)
  (last quiz)
  ↳ Covering up to 15.2

* **HW 9:** Due Monday, Dec 7.
  (last HW)

* **Final Exam:** Take-Home. Available: Sunday, Dec 13 @ 2:30pm PST.
  Due: Friday, Dec 18 @ 2:30pm PST.

$X : \Omega \to \mathbb{R}$ r.v. $\quad F_X$ CDF $\quad c > 0.$

$G_c : t \mapsto F_X(t+c) - F_X(t)$ is $L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda).$

&

$$\boxed{\int_{\mathbb{R}} (F_X(t+c) - F_X(t))\, \lambda(dt) = c.}$$

$F_X \uparrow$

$G_c \geq 0$

$$\int G_c = c < \infty$$

$$F_X(t+c) = \mathbb{P}(X \leq t+c)$$

$$- \quad F_X(t) = \mathbb{P}(X \leq t)$$

$$G_c(t) = \mathbb{P}(X \in (t, t+c]) = \mu_X(t, t+c].$$

$$\therefore \int_{\mathbb{R}} G_c\, d\lambda = \int_{\mathbb{R}} d\lambda \int d\mu_X(\omega) \mathbb{1}_{(t,t+c]}(\omega) = \int_{\mathbb{R}} \mathbb{1}_{(t,t+c]}\, d\mu_X$$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{R}} d\mu_X \int_{\mathbb{R}} d\lambda\, \mathbb{1}_{(t,t+c]} = \int c\, d\mu_X = c.$$

$X \in L^1 \Rightarrow X \in L^2$

$X \in L^2 \Longrightarrow X \in L^1$

$\|X\|_{L^1} \leq \|X\|_{L^2}$

$\|X\|_{L^2} \leq \|X\|_{L^1}$

If $\|X\|_{L^1} \leq \|X\|_{L^2}$

Then if $X \in L^2$,

then $\|X\|_{L^1} \leq \|X\|_{L^2} < \infty$

$\Rightarrow X \in L^1$.

$L^1 \supseteq L^2 \supseteq L^{\pi} \supseteq L^{10} \supseteq \cdots$

$\|X\|_{L^1} = \mathbb{E}[|X|]$ C-S

$= \mathbb{E}[|X \cdot 1|] \leq \|X\|_{L^2} \|1\|_{L^2}$

$\mathbb{E}[1^2] = 1$.

$\|X\|_{L^p} \leq \|X\|_{L^q}$ if $p \leq q$.

i.e. $L^p \supseteq L^q$ if $p \leq q$ (Hölder's inequality)

$$\ell^p = \left\{ (x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} = L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu)$$

$\uparrow$ counting measure.

$\ell^p \subseteq \ell^q$ if $p \le q$.

$\ell^1 \subseteq \ell^2 \subseteq \ell^3$

Doob-Dynkin:

Iff Y is $\sigma(X_1, \dots, X_n) / \mathcal{B}(\mathbb{R})$ - meas.

then $Y = F(X_1, \dots, X_n)$ $\exists F : \mathbb{R}^n \to \mathbb{R}$

$\mathcal{B}(\mathbb{R}^n) / \mathcal{B}(\mathbb{R})$ - meas.

If $Y = f(X_1, \dots, X_n)$ $f : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{B}(\mathbb{R}^n) / \mathcal{B}(\mathbb{R})$,

$\Rightarrow Y$ is $\sigma(\underbrace{X_1, \dots, X_n}_{X})$ - meas.

$$Y^{-1}(\underset{\mathcal{B}(\mathbb{R})}{\underset{\in}{B}}) = f(X)^{-1}(B) = X^{-1}(\underbrace{f^{-1}(B)}_{\in \mathcal{B}(\mathbb{R}^n)})$$

$\therefore \sigma(Y) \subseteq \sigma(X_1, \dots, X_n)$