MATH 280A: Probability Theory I

Tuesday, Nov 24, 2020

* **Video Lectures:** 14.1, 14.2, 14.3, 15.1, 15.2 posted on YouTube
* **HW 8:** Due Monday, Nov 30. (4 problems, #3-4 pretty short)
* **Quiz 5:** Next Thursday, Dec 3
  ↳ Covering up to 15.2
* **Thanksgiving:** Have a nice break!
  Please stay safe.

Quiz 4, Question 9: $\Omega = \mathbb{R}$

True or False?

μ, ν on $(\Omega, \mathcal{F})$ $\quad \mu \ll \nu$ & $\nu \ll \mu$.

$\Rightarrow$? $\cancel{\mu = C \cdot \nu}$. $(C > 0)$.

$\Rightarrow d\mu = f_1 \, d\nu$ where $f_1 > 0$ a.s. $[\nu]$ = a.s. $[\mu]$

$\mu \ll \nu \Leftrightarrow \frac{d\mu}{d\nu}$ exists $\qquad \nu \ll \mu \Leftrightarrow \frac{d\nu}{d\mu}$ exists

$\frac{d\mu}{d\nu} = f_1 \qquad \frac{d\nu}{d\mu} = f_2$

$d\mu = f_1 \, d\nu \qquad d\nu = f_2 \, d\mu$.

≥ 0, measurable. $\qquad \geq 0$, measurable

$d\mu = f_1 \, d\nu = f_1 f_2 \, d\mu$

$g \in C_c \quad \Rightarrow g f_1 f_2 = g$ a.s. $[\mu]$

I.e. $\mu(B) = \int_\Omega f_1 f_2 \, d\mu$; $\int g \, d\mu = \int g f_1 f_2 \, d\mu$. $\quad f_1 f_2 = 1$ a.s. $[\mu]$

HW7 Problem 2:

$$\left.\begin{array}{l} X_n - a_n \xrightarrow[\mathbb{P}]{} 0 \\ a_n \to a \end{array}\right\} \Rightarrow X_n \xrightarrow[\mathbb{P}]{} a.$$

$$\mathbb{P}\left(|a_n - a| \geq \tfrac{\varepsilon}{2} \right) = \begin{cases} 1 & \text{if } |a_n - a| \geq \varepsilon/2 \\ 0 & \text{if } |a_n - a| < \varepsilon/2 \end{cases}$$

for all large n.

Quiz 4, Problem 6.

RV X, F_X.

μ_X has a density wrt λ iff:

✗ F_X is right-continuous

Maths-off might think so → ✗ F_X is continuous

✗ F_X is a.s. diff-ble   singular continuous (e.g. Devil's staircase)

✓ $\underline{F_X \in C^1}$.

↳ $f_X = F_X' \in C(\mathbb{R})$

HW7, Problem 5: $\mathcal{A} \subset \mathcal{F}$

$\sigma(\mathcal{A}) = \mathcal{F}$

φ $\mathcal{A}$-simple $\varphi^{-1}\{t\} \in \mathcal{A}$. $\forall t \in \mathbb{R}$

$\varphi = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{A_n}$ $A_n \in \mathcal{A}$

If $X \in B(\Omega, \mathcal{F})$, $\forall \varepsilon > 0$, $\exists$ $\mathcal{A}$-simple φ s.t. $\mathbb{E}[|X-\varphi|] < \varepsilon$.

Use Dynkin's Mult Systems Thm.

$\mathbb{M} = \{\mathcal{A}\text{-simple functions}\} \ni \varphi, \psi$

$\subset$ mult.

$\varphi = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{A_n}$

$\psi = \sum_{m=1}^{M} \beta_m \mathbb{1}_{B_m}$

$$\boxed{\begin{aligned}&\mathbb{E}[|X+Y-\varphi+\psi|]\\ &\le \mathbb{E}[|X-\varphi|] + \mathbb{E}[|Y-\psi|]\end{aligned}}$$

$\varphi\psi = \sum_{n,m} \alpha_n \beta_m \mathbb{1}_{A_n \cap B_m}$

$\cap \mathcal{A}$

in

$1 \in \mathbb{H}$

$=$

$\{X \in B(\Omega, \mathcal{F}) : \forall \varepsilon > 0\ \exists \varphi \in \mathcal{A}\text{-simple}$ s.t. $\mathbb{E}[|X-\varphi|] < \varepsilon\}$.

$\therefore \sigma(\mathcal{A}) \subset \mathcal{F}$

- $1 \in \mathbb{H}$ ✓
- $\mathbb{H}$ is a subspace ✓
- $\mathbb{M} \subseteq \mathbb{H}$ ✓
- $\mathbb{H}$ is closed under bdd conv.

$\Rightarrow B(\Omega, \sigma(\mathbb{M})) \subset \mathbb{H}$.

$\mathbb{M} \ni \mathbb{1}_A$

$B(\Omega, \mathcal{F})$

$\sigma(\mathbb{M}) \ni A \quad \forall A \in \mathcal{A}$

$$\left.\begin{array}{l} X_n \in \mathbb{H}, \quad |X_n| \le M \quad \forall n. \\ X_n \to X \text{ a.s.} \end{array}\right\} \Rightarrow \text{By DCT, } X_n \to X \text{ in } L^1.$$

$$E[|X_n - X|] \to 0.$$

$\cancel{E[|X_n - \varphi_n|] < \varepsilon}$?

φ_n A-simple.

$\therefore$ Find n s.t. $E[|X_n - X|] < \frac{\varepsilon}{2}$.

Now find A-simple φ s.t.

$$E[|X_n - \varphi|] < \frac{\varepsilon}{2}$$

$X_n \in \mathbb{H}$.

$$\therefore E[|X - \varphi|]$$
$$= E[|X - X_n + X_n - \varphi|]$$
$$\le E[|X - X_n|] + E[|X_n - \varphi|] < \varepsilon.$$

$\therefore \mathbb{H}$ is closed under bdd convergence.