MATH 280A: Probability Theory I

Tuesday, Nov 17, 2020

* **Video Lectures:** 12.1, 12.2, 13.1, 13.2, 14.1, 14.2 posted on YouTube
* **Quiz 4:** This Thursday, Nov 19.
  - ↳ Covering up to 12.2 (I mean it this time!)
  - ↳ 1-1:50pm or 7-7:50pm (Google Form)
* **HW 7:** Due Monday, Nov 23, by 9pm.
* **Thanksgiving:** Next week, Lectures 15.1-15.2 will be posted by Tuesday; no lectures will be posted during Thanksgiving. HW 8 will be shortened; still due on Monday, Nov 30.

If $X \sim \mathcal{N}(0,1)$, $f \in C^1$ s.t. $f'(X),\ Xf(X) \in L^1(\mathbb{P})$

If $f \in C^1_c(\mathbb{R})$ $\longrightarrow$ $\mathbb{E}[f'(X)] = \mathbb{E}[Xf(X)]$

$$\int_{\mathbb{R}} f'(x) \underbrace{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{g} dx \qquad \qquad \int x f(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$0 \le \psi_n \le 1$ $\quad C^1(\mathbb{R})$

$\psi_n = 1$ on $[-n,n]$

$\psi_n(x) = 0$ if $|x| \ge 2n$

$(\psi_n f) \in C^1_c(\mathbb{R})$

$\parallel$

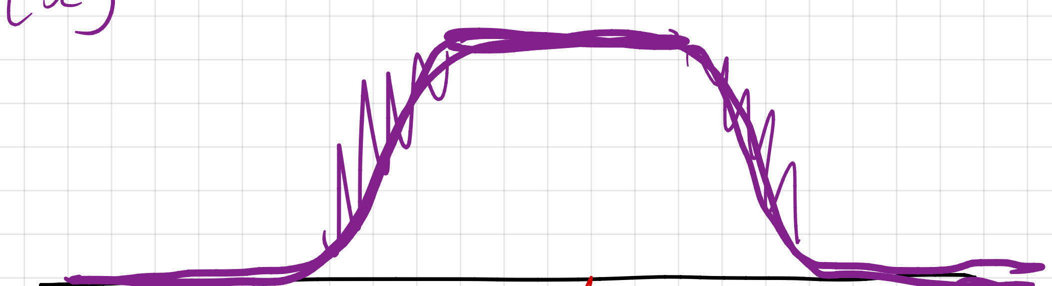
f on $[-n,n]$

$f, g \in C^1[-M,M]$ s.t. $f(\pm M) = g(\pm M) = 0$! $\longrightarrow$ $\displaystyle\int_{-M}^{M} f'g = -\int_{-M}^{M} fg'$

$\psi_n(x) = \psi(x/n)$

$\psi(x) = \psi_1(x)$

$\therefore \lim_{M\to\infty} (\quad) = \lim_{M\to\infty} (\quad)$



$\psi_n'(x) = \frac{1}{n}\psi'(x/n)$

$\forall n$ $\quad \lim_{n\to\infty} \mathbb{E}[(f\psi_n)'(X)] = \overset{\lim_{n\to\infty}}{\mathbb{E}}[Xf\psi_n(X)]$

$[-n,n]$ $\quad \overset{0}{\cancel{\psi_n'}} f + \underset{1}{\psi_n} f'$

$(\psi_n f)' \longrightarrow f'$ on $\mathbb{R}$

$(\psi_n f)'(X) \longrightarrow f'(X)$

$$\int g\,d\lambda = \lim_{M\to\infty}\int_{-M}^{M} g(x)\,dx.$$

$$\therefore\ j(M) = \int_{-M}^{M} g(x)\,dx$$

$\uparrow$ convergent.

$\therefore$ Cauchy.

$$j(M) - j(N) \to 0 \text{ as } M, N \to \infty.$$

$$\therefore\ j(2M) - j(M) \to 0$$

$$\int_{M}^{2M+\varepsilon''} g(x)\,dx + \int_{-2M-\varepsilon}^{-M} g(x)\,dx \to 0,$$

$$E[f'(X)] = E[Xf(X)]$$

$$\int_{-\infty}^{\infty} f'(x)\rho(x)\,dx \overset{?}{=} \int_{-\infty}^{\infty} x f(x)\rho(x)\,dx$$

$$\lim_{M\to\infty}\int_{-M}^{M} f'(x)\rho(x)\,dx \overset{?}{=} \lim_{M\to\infty}\int_{-M}^{M} x f(x)\rho(x)\,dx$$

$$\int_{-M}^{M} f'(x)\rho(x)\,dx = f(M)\rho(M) - f(-M)\rho(-M) - \int_{-M}^{M} x f(x)\rho(x)\,dx$$

$$(f(M) - f(-M))\,\rho(M)$$

$$\limsup_{M\to\infty} |f(M)\rho(M)| = 0.$$

$$f(X) \in L^1(P)$$

$$g \in C[a,b],\quad \int_a^b g(x)\,dx = g(c)(b-a) \text{ for some } c \in [a,b]$$

$$\therefore\ \exists\, c \in [M, M+\varepsilon] \text{ s.t.}$$

$$\int_M^{M+\varepsilon} g = g(c_M)\,\varepsilon. \quad \downarrow 0.$$

$$Xf(X) \in L^1(P)$$

$$x f(x) \in L^1(\mathbb{R}, \rho(x)\,dx)$$

$$\int_{-\infty}^{\infty} |x|\,|f(x)|\,\rho(x)\,dx < \infty$$

Fact: $g \in L^1(\lambda)$, $g \in C(\mathbb{R})$, $\lim_{x\to\pm\infty} g(x) = 0$.

$\omega \in \{A_n \text{ i.o. } [n]\} \iff \exists$ infinitely many n s.t. $\omega \in A_n$.

$\omega \in (A_n \text{ i.o.})^c \iff \exists N$ s.t. $\omega \in A_n^c \ \forall n \geq N$

$$(A_n \text{ i.o.}) = \bigcap_{k \geq 1} \underbrace{\bigcup_{n \geq k} A_n}_{\omega}$$

$\omega \iff$ for some $n \geq k$, $\omega \in A_n$.