MATH 280A: Probability Theory I

Tuesday, Nov 11, 2020 [Special Time: 11-11:50am]

* **Video Lectures:** 10.1, 10.2, 11.1, 11.2 posted on YouTube

* **Quiz 3:** Two problems mistakenly covered material past the cutoff for the quiz (on DCT). Both were removed after grading.

* **HW 6:** Due Monday, Nov 16, by 9pm.

HW5 #1.

X_n $\quad \sum_{n=1}^{\infty} E[|X_n|] < \infty \Rightarrow \lim_{n\to\infty} X_n = 0$ a.s. $\quad \forall k \in \mathbb{N}$

$\forall \cancel{\varepsilon > 0}\, k \; \exists N$ s.t. $\forall n \geq N \; |X_n| < \cancel{\varepsilon}\, \frac{1}{k}$.

$$P\left(|X_n| \geq \cancel{\varepsilon}\, \tfrac{1}{k}\right) < \frac{E[|X_n|]}{\cancel{\varepsilon}\, 1/k}$$

Markov

$$\therefore \sum_{n=1}^{\infty} P(|X_n| \geq \varepsilon) \leq \sum_{n=1}^{\infty} \frac{E[|X_n|]}{\varepsilon} < \infty$$

Borel-Cantelli

$\Rightarrow \quad P(|X_n| \geq \varepsilon$ for infinitely many $n) = 0$.

i.e. $\quad P(\{|X_n| \geq \varepsilon$ for only finitely many $n\}) = 1$.

I.e. there is some largest N s.t. $|X_n| \geq \varepsilon$ holds.

Quiz 3

Q8

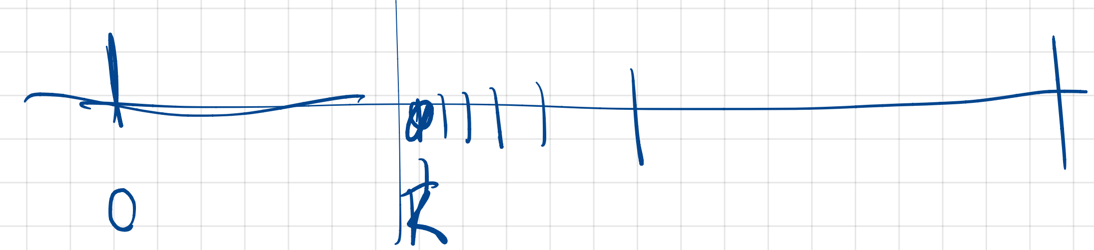
$f:[0,1]\to\mathbb{R}$

$f(\frac{1}{n})=n$, $n\in\mathbb{N}$

$f(x)=0$ if $x\in[0,1]$, $x\neq\frac{1}{n}$ for any $n\in\mathbb{N}$.

simple.

$$f_k(x)=\begin{cases}0 & \text{if } x\neq\frac{1}{n} \text{ for any } n\\ n & \text{if } x=\frac{1}{n},\ n<k\\ 0 & \text{if } x=\frac{1}{n},\ n>k.\end{cases}$$



$f_k\uparrow f$ $\therefore$ by MCT, $\int f_k\,d\lambda\uparrow\int f\,d\lambda \stackrel{?}{=} 0$.

$\|$

$f_k=0$ a.s. $[\lambda]$

$0=\int 0\,d\lambda$

HW5 #5

Eg. δ_0 is a Radon measure

$\delta_0(B) = \begin{cases} 0 & \text{if } 0 \notin B \\ 1 & \text{if } 0 \in B . \end{cases}$

CDF $= \mathbb{1}_{[0,\infty)}$.

$$F(x) = \int_{-\infty}^{x} f(t)\, \delta_0(dt) = \begin{cases} f(0) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 . \end{cases}$$

$$\parallel$$

$$\int f \, \mathbb{1}_{[0,x]} \, d\delta_0$$

$$= \left(f \, \mathbb{1}_{[0,x]}\right)(0)$$

Unless $f(0) = 0$, even continuous $f \rightsquigarrow F$ is discontinuous @ 0.

HW5 #4:

$$\nu(A) = \int_A \rho \, d\mu .$$

$f \geq 0$

$$\int f \, d\nu = \sup\left\{ \int \varphi \, d\nu : \varphi \leq f, \varphi \text{ simple} \right\}$$

$$\varphi = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j}$$

$$\int \varphi \, d\nu = \sum_{j=1}^{n} \alpha_j \nu(A_j) = \sum_{j=1}^{n} \alpha_j \int_{A_j} \rho \, d\mu .$$

$$= \sum_{j=1}^{n} \alpha_j \int \rho \mathbb{1}_{A_j} d\mu$$

$$= \int \left(\sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j} \right) \rho \, d\mu .$$

$$= \int \varphi \rho \, d\mu .$$

$$\therefore \int f \, d\nu = \sup\left\{ \int \varphi \rho \, d\mu : \varphi \leq f, \varphi \text{ simple} \right\}$$

$$= \lim_{n \to \infty} \int \underbrace{\varphi_n \rho}_{} \, d\mu$$

Close φ_n's. s.t. $\varphi_n \uparrow$.

by MCT

$$= \int f \rho \, d\mu .$$

(c) $f \in L^1(\nu) \iff |f|\rho \in L^1(\mu)$, $\int f \, d\nu = \int f \rho \, d\mu$.

$$\Updownarrow$$

$$\int |f| \, d\nu < \infty$$

$$\|$$

$$\int |f| \rho \, d\mu < \infty$$

$$\Updownarrow$$

$$|f|\rho \in L^1(\mu)$$

$$\int f \, d\nu = \int f_+ \, d\nu - \int f_- \, d\nu$$

$$\overset{(b)}{=} \int f_+ \rho \, d\mu - \int f_- \rho \, d\mu$$

$$f_\pm \leq |f|$$

$$f_\pm \rho \leq |f|\rho \in L^1(\mu)$$

$$= \int (f_+ \rho - f_- \rho) \, d\mu$$

$$= \int (f_+ - f_-) \rho \, d\mu = \int f \rho \, d\mu.$$