MATH 280A: Probability Theory I

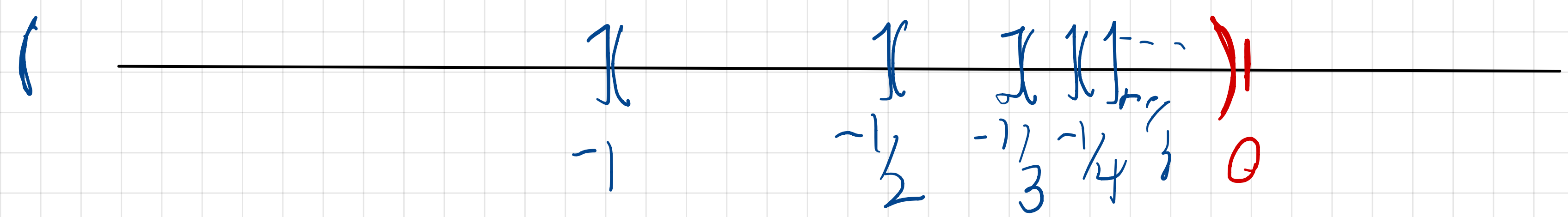
Tuesday, October 13, 2020

* **Video Lectures**: 3.1, 3.2 posted on YouTube
* **Quiz 1**: Grades released on Gradescope
  ↳ Regrade Request window now open, until tomorrow 8pm
  ↳ Q11 had a grading error; now corrected.
* **HW2**: Due Monday, October 19, 9pm.
* **Lecture 2.2**: Clarifications about finite additivity
  over semi-algebras.

Q11: $\mathcal{C} := \{$Countable disjoint unions of half open intervals $(a,b]\}$ $-\infty \le a \le b \le \infty$

Field?
- $\emptyset \in \mathcal{C}$ ✓
- if $A_1, A_2, \ldots, A_n \in \mathcal{C}$ then $A_1 \cup \cdots \cup A_n \in \mathcal{C}$ ✓
- if $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$.

$$(-\infty,-1] \cup \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, -\frac{1}{n+1}\right] = (-\infty, 0) \in \mathcal{C}$$

Eg. 

$(-\infty, 0)^c = [0, \infty)$

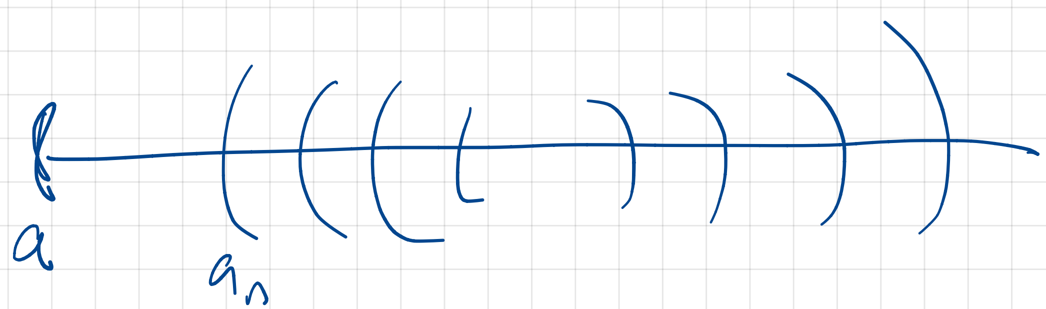
Not in $\mathcal{C}$

$\overline{E_1 \cup E_2 \cup E_3 \cup \cdots}$

$U_n = E_1 \cup \cdots \cup E_n$

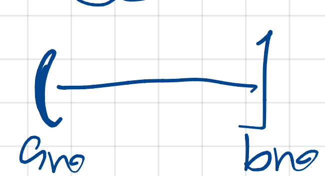
$\bigcup_{j=1}^{n} E_j = \bigcup_{j=1}^{n} U_j \not\ni a \quad \forall n.$

$\inf U_n = a_n$

$U_n \uparrow$ 

$a_n \downarrow$

$a_n \not\to -\infty$, or $a_n \downarrow a > -\infty$

if $a_{n_0} = a$, 

Lecture 2.1: Clarifications.

Semi-algebra $\mathcal{S} \subset 2^{\Omega}$

- $\phi \in \mathcal{S}$
- $\mathcal{S}$ is closed under finite intersections.
- If $E \in \mathcal{S}$ then $E^c = \bigsqcup_{j=1}^{n} A_j \quad A_j \in \mathcal{S}$.

$\mathcal{S}$ is <u>not</u> nec. closed under finite unions.

If $\mathcal{A}$ is closed under finite unions, and I want to verify a property like $\mu(\bigsqcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j) \quad \forall n,$

$\iff \quad \mu(A \sqcup B) = \mu(A) + \mu(B) \quad \forall A, B \in \mathcal{A}.$ <u>induction</u>

Def: $\chi : \mathcal{S} \to [0,\infty]$ is <u>finitely additive</u> if.

$$\chi\Big(\underbrace{\bigsqcup_{j=1}^{n} E_j}_{\in\, \mathcal{S}}\Big) = \sum_{j=1}^{n} \chi(E_j)$$ whenever $E_1, \dots, E_n$ in $\mathcal{S}$ and $\bigsqcup_{j=1}^{n} E_j$;

$$\Downarrow \qquad \not\Uparrow$$

$$\chi(E \sqcup F) = \chi(E) + \chi(F)$$ whenever $E, F, E \sqcup F \in \mathcal{S}$

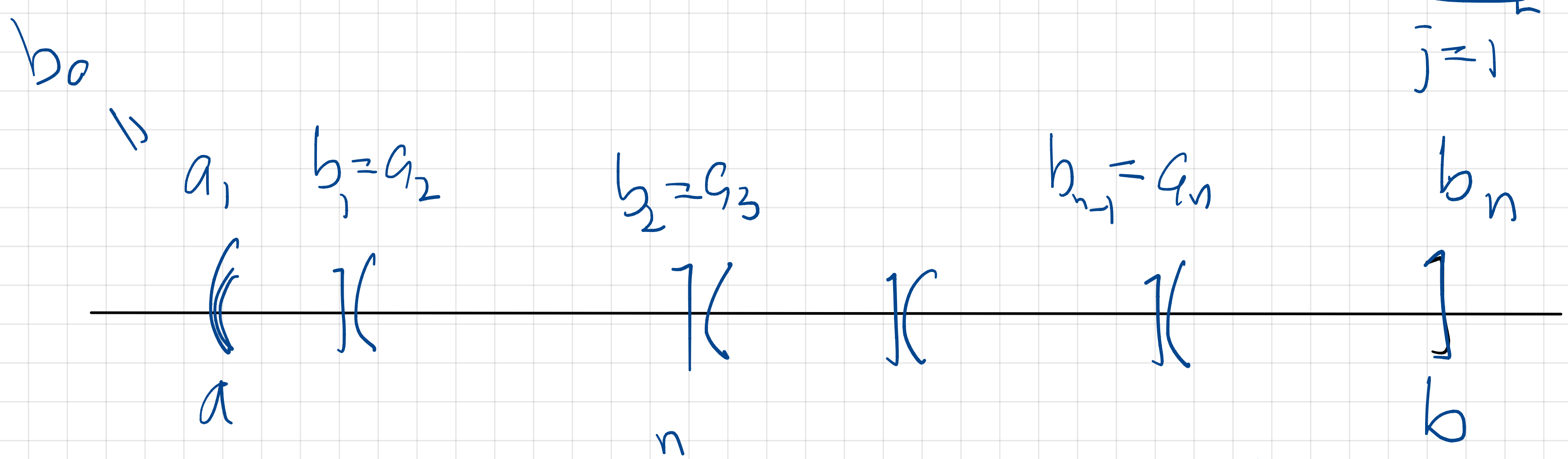
$$\lambda_F : \mathcal{B}_{(]} \to [0,\infty)$$

$$\lambda_F(a,b] = F(b) - F(a)$$

additive on $\mathcal{A}_{(]}$

I.e. $(a,b] = \bigsqcup_{j=1}^{n} (a_j, b_j] \Rightarrow F(b) - F(a)$

$$= \sum_{j=1}^{n} \left[F(b_j) - F(a_j) \right]$$



up to relabeling

$$\sum_{j=1}^{n} \left[F(b_j) - F(a_j) \right] = \sum_{j=1}^{n} \left[F(b_j) - F(b_{j-1}) \right] = F(b_n) - F(b_0)$$

$$= F(b) - F(a).$$

telescoping

Q9: T or F: If Ω is countable, every field over Ω is a σ-field.

HW1 #2: $\mathcal{F} = \{E \subseteq \Omega \text{ s.t. } E \text{ or } E^c \text{ is finite}\}$ is a field.

$\hookrightarrow E \cup F$

$|E| < \infty$ $\quad |F^c| < \infty$

$(E \cap F)^c = E^c \cap F^c \subseteq F^c$

$\therefore$ finite. $\quad$ finite

NOT a σ-field.

$\Omega = \mathbb{N}$, $\quad \Theta = \{1, 3, 5, 7, 9, \ldots\} \notin \mathcal{F} \quad \Theta^c = \{2, 4, 6, 8, \ldots\}$

$= \{1\} \cup \{3\} \cup \{5\} \cup \{7\} \cup \cdots$

$= \bigcup_{j=1}^{\infty} \{2j-1\} \in \sigma(\mathcal{F}) \ni \Theta$