Premeasures, Finitely-Additive Measures (II. 5.2.1 in Driver)

Genuine measures (defined on full σ-fields) are often difficult to construct, owing to all the wild sets in a σ-field.

To approach this problem, we often start with weaker notions of "measure" that we later build up to the full deal.

Def: A pair $(\Omega, \mathcal{A})$ is a premeasurable space if $\mathcal{A}$ is a ~~σ~~ field over Ω.
A countably additive function $\mu : \mathcal{A} \to [0, \infty]$ is a **premeasure**.

If we assume μ is only finitely-additive

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

we call it a **finitely-additive measure**.

Proposition: Let $(\Omega, \mathcal{A}, \chi)$ be a finitely-additive measure space.

If $\{A_i\}_{i=1}^{\infty}$ are disjoint in $\mathcal{A}$, and it so happens that $A = \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$,

then $\chi(A) \geq \sum_{i=1}^{\infty} \chi(A_i)$.

Pf. Finitely-additive measures are monotone (proof identical to the 'measure' case).

A "Borel Field" $\mathcal{B}_{(\,]}(\mathbb{R})$

Among the many natural generating sets for the Borel σ-field $\mathcal{B}(\mathbb{R})$ is

$$\mathcal{I}_{(\,]} = \{ (a,b] : -\infty \leq a \leq b \leq \infty \}$$

What about the field generated by these intervals?

$$\mathcal{A}(\mathcal{I}_{(\,]}) \supseteq \{ \text{finite unions of intervals in } \mathcal{I}_{(\,]} \}$$

Prop:

Semi-Algebras of Sets

A collection $\mathcal{S} \subseteq 2^{\Omega}$ is a **semi-algebra** or **elementary class** if

(1) $\emptyset \in \mathcal{S}$

(2) If $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$

(3) If $A \in \mathcal{S}$ then A^c is a finite disjoint union of elements from $\mathcal{S}$.

Prop: If $\mathcal{S}$ is a semi-algebra over Ω, then the field $\mathcal{A}(\mathcal{S})$ it generates is equal to

$$\mathcal{A}(\mathcal{S}) = \{\text{all finite disjoint unions of sets from } \mathcal{S}\}$$

Prop: If $\mathcal{S}$ is a semi-algebra over Ω, then $A(\mathcal{S})$ is equal to
$DU(\mathcal{S}) := \{$ all finite disjoint unions of sets from $\mathcal{S}\}$

Pf. $\mathcal{S} \subseteq DU(\mathcal{S}) \subseteq A(\mathcal{S})$

$\therefore$ Suffices to show that $DU(\mathcal{S})$ is a field.

* Closure under finite $\cap$:

Let $D = \bigsqcup_{i=1}^{n} A_i$, $E = \bigsqcup_{j=1}^{m} B_i \in DU(\mathcal{S})$.

Then $D \cap E = \bigcup_{\substack{1 \le i \le n \\ 1 \le j \le m}} A_i \cap B_j$

* Closure under complement:

$D^c = \bigcap_{i=1}^{n} A_i^c$

Finitely Additive Measures and Semi-Algebras

Prop: Let $\mathcal{S}$ be a semi-algebra over Ω.

Let $\chi : \mathcal{S} \to [0,\infty]$ be finitely additive: $\chi(E \cup F) = \chi(E) + \chi(F)$, $E, F \in \mathcal{S}$.

Then χ extends to a unique finitely-additive measure on $\mathcal{A}(\mathcal{S})$, defined by

$$A = \bigsqcup_{i=1}^{n} E_i \Rightarrow \chi(A) := \sum_{i=1}^{n} \chi(E_i).$$

Pf. This formula must hold if χ is a f.a. measure. It $\therefore$ uniquely defines the extended χ; and it is routine to check finite additivity.

The main issue is to show it is well-defined:

$$A = \bigsqcup_{i=1}^{n} E_i = \bigsqcup_{j=1}^{m} F_j$$

Stieltjes (pre) Measures on $\mathcal{B}_{(]}(\mathbb{R})$

Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function.

On the semi-algebra $\mathcal{S}_{(]} = \{(a,b] : -\infty \le a \le b \le \infty\}$, define

$$\lambda_F((a,b]) =$$

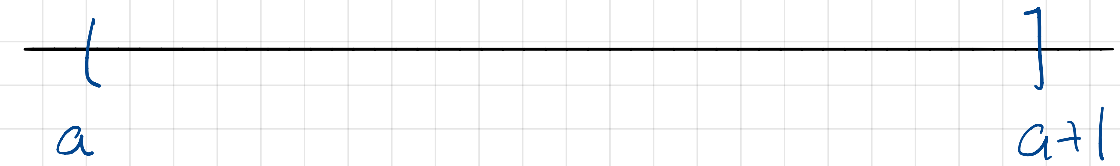
This is additive on the semi-algebra $\mathcal{S}_{(]}$:

$\therefore$ λ_F extends to a finitely-additive measure on $\mathcal{A}(\mathcal{S}_{(]}) = \mathcal{B}_{(]}(\mathbb{R})$.

But: is it a premeasure? Is it countably additive?

E.g. Fix $a \in \mathbb{R}$. $\quad (a, a+1] \in \mathcal{B}_{(\,]}(\mathbb{R})$

$$(a, a+1] = \bigsqcup_{n=1}^{\infty} \left(a + \frac{1}{n+1}, a + \frac{1}{n}\right]$$



$$\chi_F\big((a, a+1]\big) = F(a+1) - F(a)$$

$$\sum_{n=1}^{\infty} \chi_F\left(\left(a + \frac{1}{n+1}, a + \frac{1}{n}\right]\right) = \sum_{n=1}^{\infty} \left(F\left(a + \frac{1}{n}\right) - F\left(a + \frac{1}{n+1}\right)\right)$$