Premeasures, Finitely-Additive Measures (II. 5.2.1 in Driver)

Genuine measures (defined on full σ-fields) are often difficult to construct, owing to all the wild sets in a σ-field.

To approach this problem, we often start with weaker notions of "measure" that we later build up to the full deal.

Def: A pair $(\Omega, \mathcal{A})$ is a premeasurable space if $\mathcal{A}$ is a ~~σ~~-field over Ω.

A countably additive function $\mu : \mathcal{A} \to [0,\infty]$ is a **premeasure**.

If $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ s.t. $\bigsqcup_{i=1}^{\infty} E_i = E \in \mathcal{A}$ then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$

If we assume χ is only finitely-additive

$$\chi(A \sqcup B) = \chi(A) + \chi(B)$$

we call it a **finitely-additive measure**.

Proposition: Let $(\Omega, \mathcal{A}, \chi)$ be a finitely-additive measure space.

If $\{A_i\}_{i=1}^{\infty}$ are disjoint in $\mathcal{A}$, and it so happens that $A = \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$,

then $$\underline{\chi(A)} \geq \sum_{i=1}^{\infty} \chi(A_i).$$ ← CAN be strict.

Pf. Finitely-additive measures are monotone (proof identical to the 'measure' case).

Fix $n \in \mathbb{N}$, $\bigsqcup_{i=1}^{n} A_i \in \mathcal{A}$ $\subseteq A$

$$\therefore \chi\left(\bigsqcup_{i=1}^{n} A_i\right) \leq \chi(A)$$

$$\| $$

$$\sum_{i=1}^{n} \chi(A_i) \qquad \lim_{n\to\infty}$$

A "Borel Field" $\mathcal{B}_{(]}(\mathbb{R})$

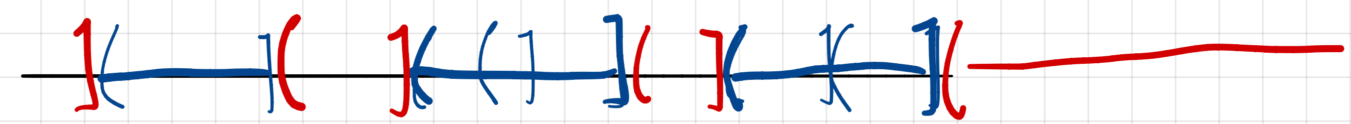
Among the many natural generating sets for the Borel σ-field $\mathcal{B}(\mathbb{R})$ is

$$\mathcal{C}_{(]} = \{ (a,b] \ : \ -\infty \le a \le b \le \infty \}$$

$$(a,b] = \{ x \in \mathbb{R} : a < x \le b \}$$
$$(a,a] = \emptyset$$
$$(a,\infty] = (a,\infty)$$

What about the field generated by these intervals?

$$\mathcal{A}(\mathcal{C}_{(]}) \ \cancel{\overset{?}{=}}\ \underline{=} \ \{ \text{finite disjoint unions of intervals in } \mathcal{C}_{(]} \} = \mathcal{B}_{(]}(\mathbb{R})$$



Prop: Check that $\mathcal{B}_{(]}(\mathbb{R})$ is a field.

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Semi-Algebras of Sets

A collection $\mathcal{S} \subseteq 2^{\Omega}$ is a **semi-algebra** or **elementary class** if

(1) $\emptyset \in \mathcal{S}$

(2) If $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$

(3) If $A \in \mathcal{S}$ then A^c is a finite disjoint union of elements from $\mathcal{S}$.

$\Omega = \emptyset^c$ is a finite disjoint union of elements in $\mathcal{S}$.

Prop: If $\mathcal{S}$ is a semi-algebra over Ω, then the field $\mathcal{A}(\mathcal{S})$ it generates is equal to

$$\mathcal{A}(\mathcal{S}) = \{\text{all finite disjoint unions of sets from } \mathcal{S}\}$$

Prop: If $\mathcal{S}$ is a semi-algebra over Ω, then $\mathcal{A}(\mathcal{S})$ is equal to

$DU(\mathcal{S}) := \{$ all finite disjoint unions of sets from $\mathcal{S}\}$

closed under finite union

Pf. $\mathcal{S} \subseteq DU(\mathcal{S}) \subseteq \mathcal{A}(\mathcal{S})$

$\therefore$ Suffices to show that $DU(\mathcal{S})$ is a field.

* Closure under finite $\cap$:

Let $D = \bigsqcup_{i=1}^{n} A_i$, $E = \bigsqcup_{j=1}^{m} B_j \in DU(\mathcal{S})$.

Then $D \cap E = \bigcup_{\substack{1 \le i \le n \\ 1 \le j \le m}} A_i \cap B_j \in DU(\mathcal{S})$

Claim: all $A_i \cap B_j$ are disjoint for diff. (i,j)

* Closure under complement:

$D^c = \bigcap_{i=1}^{n} A_i^c$ ← a finite disjoint union of el'ts in $\mathcal{S}$.

$\therefore D^c \in \mathcal{S}$.

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Finitely Additive Measures and Semi-Algebras

Prop: Let $\mathcal{S}$ be a semi-algebra over Ω.

Let $\chi : \mathcal{S} \to [0,\infty]$ be finitely additive: $\chi(E \sqcup F) = \chi(E) + \chi(F)$, $E, F \in \mathcal{S}$.

Then χ extends to a unique finitely-additive measure on $\mathcal{A}(\mathcal{S})$, defined by

$$A = \bigsqcup_{i=1}^{n} E_i \Rightarrow \chi(A) := \sum_{i=1}^{n} \chi(E_i).$$

Pf. This formula must hold if χ is a f.a. measure. It $\therefore$ uniquely defines the extended χ; and it is routine to check finite additivity.

The main issue is to show it is **well-defined**:

$$A = \bigsqcup_{i=1}^{n} E_i = \bigsqcup_{j=1}^{m} F_j$$

$$E_i = \bigcup_j E_i \cap F_j$$

$$\chi(E_i) = \sum_j \chi(E_i \cap F_j)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \chi(E_i \cap F_j) = \sum_{j=1}^{m} \chi(F_j)$$

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Stieltjes (pre) Measures on $\mathcal{B}_{(]}(\mathbb{R})$

Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function.

On the semi-algebra $\mathcal{A}_{(]} = \{(a,b] : -\infty \leq a \leq b \leq \infty\}$, define

$$\chi_F((a,b]) = F(b) - F(a) \geq 0$$

This is additive on the semi-algebra $\mathcal{A}_{(]}$:

$$(a,b] = (a,c] \cup (c,b]$$

$$a < c < b$$

$$\chi_F(a,b] = F(b) - F(a)$$

$$\chi_F((a,c]) + \chi_F(c,b]$$

$$= (\cancel{F(c)} - F(a)) + (F(b) - \cancel{F(c)}) = F(b) - F(a)$$

$\therefore$ χ_F extends to a finitely-additive measure on $\mathcal{A}(\mathcal{A}_{(]}) = \mathcal{B}_{(]}(\mathbb{R})$.

But: is it a premeasure? Is it countably additive?

E.g. Fix $a \in \mathbb{R}$. $\qquad (a, a+1] \in \mathcal{B}_{(\,]}(\mathbb{R})$

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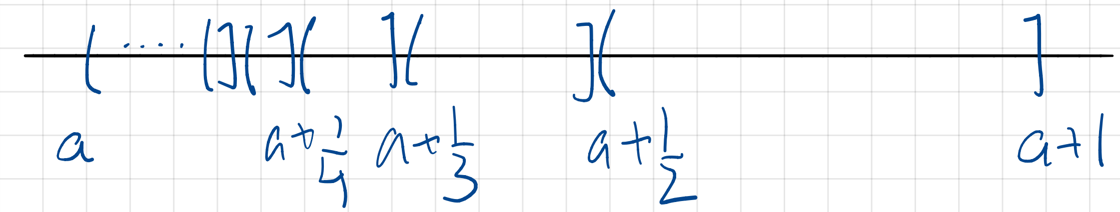
$$\bigsqcup_{n=1}^{\infty} \left(a + \frac{1}{n+1}, a + \frac{1}{n}\right]$$



$$\chi_F\big((a, a+1]\big) = F(a+1) - F(a)$$

$$\sum_{n=1}^{\infty} \chi_F\left(\left(a + \frac{1}{n+1}, a + \frac{1}{n}\right]\right) = \sum_{n=1}^{\infty} \left(F\left(a + \frac{1}{n}\right) - F\left(a + \frac{1}{n+1}\right)\right)$$

$$= \left(F(a+1) - F\left(\cancel{a + \tfrac{1}{2}}\right)\right) + \left(F\left(\cancel{a + \tfrac{1}{2}}\right) - F\left(\cancel{a + \tfrac{1}{3}}\right)\right) + \left(F\left(\cancel{a + \tfrac{1}{3}}\right) - F\left(a + \tfrac{1}{4}\right)\right) + \cdots$$

$$= F(a+1) - F\left(a + \frac{1}{m}\right) + \sum_{n=m}^{\infty} \left(F\left(a + \frac{1}{n}\right) - F\left(a + \frac{1}{n+1}\right)\right)$$

$$= F(a+1) - F(a+)$$

$$\lim_{\varepsilon \downarrow 0} F(a + \varepsilon)$$

$\therefore$ χ_F is not countably additive if $F(a+) \neq F(a)$.