Measures: Definition & Examples (II.5.1 in Driver)

$(\Omega, \mathcal{F})$ measurable space

Def:

A function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if it is countably additive. I.e.

↳ If $E_1, E_2, E_3, \ldots \in \mathcal{F}$ are all disjoint, then

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triple $(\Omega, \mathcal{F}, \mu)$ is then called a measure space.

If $\mu(\Omega) < \infty$, we say μ is a finite measure.

If $\mu(\Omega) = 1$, we say μ is a probability measure and $(\Omega, \mathcal{F}, \mu)$ is a probability space.

Eg. $\mu \equiv 0$ or $\mu \equiv \infty$ (on any $(\Omega, \mathcal{F})$).

Eg. Point mass: on $(\Omega, 2^\Omega)$, fix a point $\omega_0 \in \Omega$, and define $\delta_{\omega_0} : 2^\Omega \to \{0,1\}$ by

If $\{E_n\}_{n=1}^\infty$ are disjoint, then ω_0 is in at most 1 of the E_n.

- If $\exists n_0$ s.t. $\omega_0 \in E_{n_0}$, then $\omega_0 \in \bigsqcup_{n=1}^\infty E_n$, so $\delta_{\omega_0}\left(\bigsqcup_{n=1}^\infty E_n\right) = 1$

  $\sum_{n=1}^\infty \delta_{\omega_0}(E_n) =$

- If $\nexists n$ s.t. $\omega_0 \in E_n$, then $\omega_0 \notin \bigsqcup_{n=1}^\infty E_n$, so $\delta_{\omega_0}\left(\bigsqcup_{n=1}^\infty E_n\right) = 0$

  $\sum_{n=1}^\infty \delta_{\omega_0}(E_n) =$

New Measures from Old

Fix a measurable space $(\Omega, \mathcal{F})$.

If μ is a measure, so is $\alpha\mu$ for any $\alpha \geq 0$.

If $\{\mu_j\}_{j=1}^{\infty}$ is a countable set of measures, then

$$\mu = \sum_{j=1}^{\infty} \mu_j$$

is a measure.

Pf. If $\{E_i\}_{i=1}^{\infty}$ are disjoint events in $\mathcal{F}$, then

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{\infty} \mu_j\left(\bigsqcup_{i=1}^{\infty} E_i\right)$$

Eg. $\mu = \sum_{j=1}^{\infty} p_j \delta_{\omega_j}$ for some $\omega_j \in \Omega$, $p_j \geq 0$

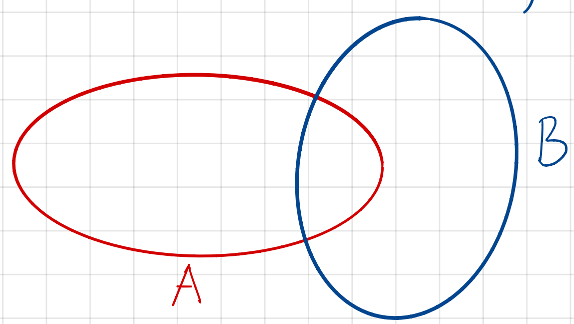
↳ If $\sum_{j=1}^{\infty} p_j = 1$, probability measure.

↳ If $\{\omega_j\}_{j=1}^{\infty}$ is all of Ω, this is discrete probability:

Basic Properties

* monotone: If $A, B \in \mathcal{F}$ and $A \subseteq B$, $\mu(A) \leq \mu(B)$.

* rule of addition: If $A, B \in \mathcal{F}$, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.



* subadditive: If $\{B_n\}_{n=1}^{\infty}$ are in $\mathcal{F}$, then $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n)$