# Measures: Definition & Examples (II.5.1 in Driver)

$(\Omega, \mathcal{F})$ measurable space  $E \in \mathcal{F}$ "measurable sets" "events"

any set, $\sigma$-field

**Def:**
A function $\mu : \mathcal{F} \to [0, \infty]$ is called a **measure** if it is countably additive. I.e.

↳ If $E_1, E_2, E_3, \ldots \in \mathcal{F}$ are all disjoint, then

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triple $(\Omega, \mathcal{F}, \mu)$ is then called a **measure space**.

If $\mu(\Omega) < \infty$, we say $\mu$ is a **finite measure**.

If $\mu(\Omega) = 1$, we say $\mu$ is a **probability measure** and $(\Omega, \mathcal{F}, \mu)$ is a **probability space**.

Eg. $\mu \equiv 0$ or $\mu \equiv \infty$ (on any $(\Omega, \mathcal{F})$).

Eg. Point mass: on $(\Omega, 2^{\Omega})$, fix a point $\omega_0 \in \Omega$, and define $\delta_{\omega_0} : 2^{\Omega} \to \{0,1\}$ by

$$\delta_{\omega_0}(E) = \begin{cases} 1 & \text{if } \omega_0 \in E \\ 0 & \text{if } \omega_0 \notin E \end{cases}$$

If $\{E_n\}_{n=1}^{\infty}$ are disjoint, then $\omega_0$ is in at most 1 of the $E_n$.

- If $\exists n_0$ s.t. $\omega_0 \in E_{n_0}$, then $\omega_0 \in \bigsqcup_{n=1}^{\infty} E_n$, so $\delta_{\omega_0}\left(\bigsqcup_{n=1}^{\infty} E_n\right) = 1$ ✓

$$\sum_{n=1}^{\infty} \delta_{\omega_0}(E_n) = 0+0+0+\cdots+\underset{n_0}{1}+0+\cdots$$

- If $\nexists n$ s.t. $\omega_0 \in E_n$, then $\omega_0 \notin \bigsqcup_{n=1}^{\infty} E_n$, so $\delta_{\omega_0}\left(\bigsqcup_{n=1}^{\infty} E_n\right) = 0$ ✓

$$\sum_{n=1}^{\infty} \delta_{\omega_0}(\underbrace{E_n}_{0}) = 0$$

# New Measures from Old

Fix a measurable space $(\Omega, \mathcal{F})$.

If $\mu$ is a measure, so is $\alpha\mu$ for any $\alpha \geq 0$. $(\alpha\mu)(E) = \alpha \cdot \mu(E)$

If $\{\mu_j\}_{j=1}^{\infty}$ is a countable set of measures, then

$$\mu = \sum_{j=1}^{\infty} \mu_j$$

$\cdot$ $\mu_j \geq 0$, allowed to be $\infty$

$\therefore$ no convergence issues.

is a measure.

Pf. If $\{E_i\}_{i=1}^{\infty}$ are disjoint events in $\mathcal{F}$, then

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{\infty} \mu_j\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} \mu_j(E_i)\right)$$

Tonnelli's theorem

$$= \sum_{i=1}^{\infty} \underbrace{\sum_{j=1}^{\infty} \mu_j(E_i)}_{\mu(E_i)}$$

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Eg. $\mu = \sum_{j=1}^{\infty} p_j \delta_{\omega_j}$ for some $\omega_j \in \Omega$, $p_j \geq 0$

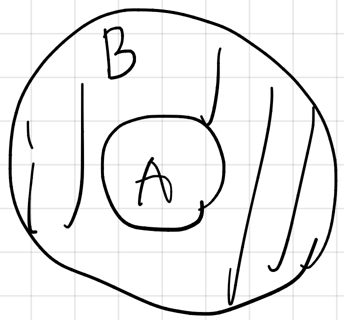
↳ If $\sum_{j=1}^{\infty} p_j = 1$, probability measure. $\mu(\Omega) = \sum_j p_j \overbrace{\delta_{\omega_j}(\Omega)}^{1} = \sum_{j=1}^{\infty} p_j = 1$.

↳ If $\{\omega_j\}_{j=1}^{\infty}$ is all of $\Omega$, this is discrete probability:

$$\mu(\underset{\mathcal{F}}{\underset{\in}{E}}) = \sum_{j:\,\omega_j \in E} p_j = \sum_{\omega \in E} \overbrace{\mu(\{\omega\})}^{p_j \text{ if } \omega = \omega_j}$$

# Basic Properties

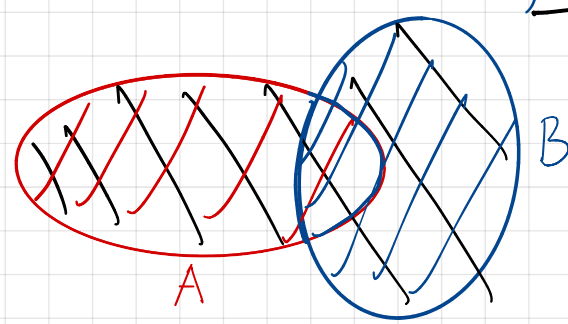
* monotone: If $A, B \in \mathcal{F}$ and $A \subseteq B$, $\mu(A) \le \mu(B)$.




$B = (B \setminus A) \sqcup A$

$\therefore \mu(B) = \underbrace{\mu(B \setminus A)}_{\geq 0} + \mu(A) \geq \mu(A)$.

* rule of addition: If $A, B \in \mathcal{F}$, $\underline{\mu(A \cup B)} + \mu(A \cap B) = \mu(A) + \mu(B)$.




$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

* subadditive: If $\{B_n\}_{n=1}^{\infty}$ are in $\mathcal{F}$, then $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu(B_n)$

$A_1 = B_1$, $A_2 = B_2 \setminus B_1$, ..., $A_n = B_n \setminus (B_1 \cup \cdots \cup B_{n-1})$

- $\bigcup_{n=1}^{N} A_n = \bigcup_{n=1}^{N} B_n \Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$
- $A_n \subseteq B_n$

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{j=1}^{\infty} \mu(A_n) \leq \sum_{j=1}^{\infty} \mu(B_n)$$