Frequentism

The SLLN justifies the "frequentist" philosophy of probability and statistics (and is the basis of all scientific experiments).

Given an event A, what does $\mathbb{P}(A)$ mean?

↳ Do repeated trials of an experiment to test A.

↳ Record 1 if A occurs in a trial, record 0 if A does not occur in a trial.

↳ Average the results.

More practically: the universality of the SLLN (depending only on $\mathbb{E}[X_n]$) makes it useful when we know little about μ_{X_n}.

Renewal Theory

Eg. $\{X_n\}_{n\geq 1}^{\infty}$ iid L^1 random variables with $X_n \geq 0$, & $\mathbb{P}(X_n > 0) > 0$.

- $\parallel$ Lifetime of lightbulb #n.
- All the same design, manufacture, so identically distributed.
- The time @ which bulb n burns out is not influenced by the lifetimes of the other bulbs – independent.
- Assume we replace each bulb the instant it burns out.

$$S_n = X_1 + X_2 + \cdots + X_n$$

Question: How many bulbs do we need to last for time t?

$$N_t :=$$

random integer determined by

Answerable question: how does N_t behave as $t \to \infty$?

"Back of an envelope" calculation:

If X_n is "not very random", then $X_n \approx X_m \ \forall n, m$,

so $S_n \approx X_1 + \cdots + X_n$

$\therefore \quad N_t = \sup\{n \in \mathbb{N} : S_n \leq t\}$

I.e. $\frac{N_t}{t}$

Prop: $\mathbb{E}[X_1] > 0$, and $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[X_1]}$.

Pf. • $X_1 \geq 0 \ \therefore \ \mathbb{E}[X_1] \geq 0$. Also, $0 = \mathbb{E}[X_1] = \|X_1\|_{L^1} \Rightarrow X_1 = 0$ a.s.

• Observe: $X_n \in L^1$, so $X_n < \infty$ a.s. $\therefore$ $S_n < \infty$ a.s.

$\Omega_1 := \bigcap_n \{X_n < \infty\}$

$N_t \to \infty$ as $t \to \infty$, on Ω_1

• By SLLN, $\Omega_0 = \{S_n/n \to \mathbb{E}[X_1]\}$ has $\mathbb{P}(\Omega_0) = 1$.

Work on $\Omega_0 \cap \Omega_1$

By definition,

$$S_{N_t} \leq t < S_{N_t+1}$$

More Realistic Example.

Lightbulbs burn out after independent iid times $X_n \geq 0$.
After each bulb burns out, there's a waiting time $Y_n \geq 0$ before it is replaced.

$$\{X_n\}_{n=1}^{\infty} \text{ iid } L^1, \quad \{Y_n\}_{n=1}^{\infty} \text{ iid } L^1$$

$$\mathbb{P}(X_n > 0) > 0, \quad \mathbb{P}(Y_n > 0) > 0.$$

Time until $(n+1)^{st}$ bulb is replaced $= (X_1+Y_1) + \cdots + (X_n+Y_n) =: S_n$

Number of bulbs needed through time $t := N_t = \sup\{n \in \mathbb{N} : S_n \leq t\}$

From the last example, we know $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[X_1+Y_1]} = \frac{1}{\mathbb{E}[X_1]+\mathbb{E}[Y_1]}$

Question: What fraction of the time is there light?

"Back of an envelope" calculation:

Time with light $= X_1 + \cdots + X_n$

Total time $= X_1 + Y_1 + \cdots + X_n + Y_n$

Lemma:

If $X \in L^1$ and $\{X_n\}_{n=1}^{\infty}$ are id with $X_n \overset{d}{=} X$,
then
$$\frac{X_n}{n} \to 0 \quad a.s.$$

Pf. We proved [Lemma 26.26] in Lecture 18.1 that
$$\sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n\varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}[|X|] \quad \text{for any } \varepsilon > 0.$$
$$\therefore \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq n\varepsilon) = \sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n\varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}[|X|]$$

$\therefore$ By Borel-Cantelli, $\mathbb{P}\left(\frac{|X_n|}{n} \geq \varepsilon \text{ i.o.}\right) = 0$

Set $\Omega_\varepsilon = \left\{ \frac{|X_n|}{n} < \varepsilon \text{ for all large } n \right\}$

$\left\{ \lim_{n\to\infty} \frac{|X_n|}{n} = 0 \right\} =$

Back to the "realistic" lightbulb problem:

$$\{X_n\}_{n=1}^{\infty} \text{ iid } L^1, \quad \{Y_n\}_{n=1}^{\infty} \text{ iid } L^1$$

$$\mathbb{P}(X_n > 0) > 0, \quad \mathbb{P}(Y_n > 0) > 0.$$

$S_n = X_1 + Y_1 + \cdots + X_n + Y_n$, $N_t = \sup\{n \in \mathbb{N} : S_n \leq t\}$

$T_t :=$ length of time in $[0,t]$ that a working lightbulb is installed.

$=$

$$\leq \quad T_t \quad \leq$$

$$\therefore \lim_{t\to\infty} \frac{T_t}{t} = \lim_{t\to\infty} \frac{\tilde{T}_t}{t} = \lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N_t} X_n$$

Conclusion:

$$\lim_{t\to\infty} \frac{T_t}{t} = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1] + \mathbb{E}[Y_1]}$$