Frequentism

The SLLN justifies the "frequentist" philosophy of probability and statistics (and is the basis of all scientific experiments).

Given an event A, what does $\mathbb{P}(A)$ mean?

- Do repeated trials of an experiment to test A.
- Record 1 if A occurs in a trial, record 0 if A does not occur in a trial.
- Average the results.

$X_n \overset{d}{=} \mathbb{1}_A$ independent.

$$\text{SLLN} \Rightarrow \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A) \quad \text{a.s.}$$

More practically: the universality of the SLLN (depending only on $\mathbb{E}[X_n]$) makes it useful when we know little about μ_{X_n}.

Renewal Theory

Eg. $\{X_n\}_{n\geq 1}^{\infty}$ iid L^1 random variables with $X_n \geq 0$, & $\mathbb{P}(X_n > 0) > 0$.

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- Lifetime of lightbulb #n.
- All the same design, manufacture, so identically distributed.
- The time @ which bulb n burns out is not influenced by the lifetimes of the other bulbs – independent.
- Assume we replace each bulb the instant it burns out.

$$S_n = X_1 + X_2 + \cdots + X_n$$

Question: How many bulbs do we need to last for time t?

$$N_t := \sup\{n \in \mathbb{N} : S_n \leq t\}$$

random integer determined by $\mu_{S_n} = \mu_{X_1}^{*n}$, t.

Answerable question: how does N_t behave as $t \to \infty$?

"Back of an envelope" calculation:

If X_n is "not very random", then $X_n \approx X_m \; \forall n,m$,

so $S_n = X_1 + \cdots + X_n \approx nX_1$

$\therefore \; N_t = \sup\{n \in \mathbb{N} : S_n \le t\} \approx \sup\{n : nX_1 \le t\} \approx \frac{t}{X_1}$.

I.e. $\frac{N_t}{t} \approx \frac{1}{X_1}$

Prop: $\mathbb{E}[X_1] > 0$, and $\lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[X_1]}$.

Pf.: $X_1 \ge 0 \;\therefore\; \mathbb{E}[X_1] \ge 0$. Also, $0 = \mathbb{E}[X_1] = \|X_1\|_{L^1} \Rightarrow X_1 = 0$ a.s. $\lightning$ $\mathbb{P}(X_n > 0) > 0$

- Observe: $X_n \in L^1$, so $X_n < \infty$ a.s. $\therefore S_n < \infty$ a.s.

  $\Omega_1 := \bigcap_n \{X_n < \infty\} \quad \mathbb{P}(\Omega_1) = 1$.

  $N_t \to \infty$ as $t \to \infty$, on Ω_1 otherwise, if $N_t \le n_0 \; \forall t > 0 \;\therefore S_{n_0+1} = \infty$. $\lightning$

$N_t \uparrow$

- By SLLN, $\Omega_0 = \{S_n/n \to \mathbb{E}[X_1]\}$ has $\mathbb{P}(\Omega_0) = 1$.

  Work on $\mathbb{P}(\Omega_0 \cap \Omega_1) = 1$.

By definition,

$$\frac{S_{N_t}}{N_t} \leq \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t}$$

$$\mathbb{E}[X_1] \underset{SLL}{=} \lim_{t\to\infty} \frac{S_{N_t(\omega)}(\omega)}{N_t(\omega)} \leq \liminf_{n\to\infty} \frac{t}{N_t}.$$

$$\limsup_{t\to\infty} \frac{t}{N_t} \leq \limsup_{t\to\infty} \frac{S_{N_t+1}}{N_t} \cdot \frac{N_t+1}{N_t+1} \qquad \frac{N_t+1}{N_t} = 1 + \frac{1}{N_t} \to 1.$$

$$= \mathbb{E}[X_1].$$

$$\therefore \lim_{t\to\infty} \frac{t}{N_t} = \mathbb{E}[X_1] > 0.$$

$$N_t / t \to \frac{1}{\mathbb{E}[X_1]}.$$

More Realistic Example.

Lightbulbs burn out after independent iid times $X_n \geq 0$.
After each bulb burns out, there's a waiting time $Y_n \geq 0$ before it is replaced.

$$\{X_n\}_{n=1}^{\infty} \text{ iid } L^1, \quad \{Y_n\}_{n=1}^{\infty} \text{ iid } L^1$$

$$\mathbb{P}(X_n > 0) > 0, \quad \mathbb{P}(Y_n > 0) > 0.$$

Time until $(n+1)^{st}$ bulb is replaced $= (X_1+Y_1) + \cdots + (X_n+Y_n) =: S_n$

Number of bulbs needed through time $t := N_t = \sup\{n \in \mathbb{N} : S_n \leq t\}$

From the last example, we know $\lim\limits_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[X_1+Y_1]} = \frac{1}{\mathbb{E}[X_1]+\mathbb{E}[Y_1]}$

Question: What fraction of the time is there light?

"Back of an envelope" calculation:

Time with light $= X_1 + \cdots + X_n \approx nX_1$

Total time $= X_1 + Y_1 + \cdots + X_n + Y_n \approx nX_1 + nY_1$

$$\left.\right\} \text{ ratio} \approx \frac{X_1}{X_1+Y_1}$$

Lemma:

If $X \in L^1$ and $\{X_n\}_{n=1}^{\infty}$ are id with $X_n \overset{d}{=} X$,

then $$\frac{X_n}{n} \to 0 \quad a.s.$$

Pf. We proved [Lemma 26.26] in Lecture 18.1 that

$$\sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n\varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}[|X|] \quad \text{for any } \varepsilon > 0.$$

$$\therefore \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq n\varepsilon) = \sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n\varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}[|X|] < \infty.$$

$$\underset{\displaystyle \mathbb{P}\left(\frac{|X_n|}{n} \geq \varepsilon\right)}{\parallel}$$

$\therefore$ By Borel-Cantelli, $\mathbb{P}\left(\frac{|X_n|}{n} \geq \varepsilon \text{ i.o.}\right) = 0$

$\mathbb{P} = 1$

Set $\Omega_\varepsilon = \left\{ \frac{|X_n|}{n} < \varepsilon \text{ for all large } n \right\}$ $\quad \mathbb{P}(\Omega_\varepsilon) = 1.$

$$\left\{ \lim_{n \to \infty} \frac{|X_n|}{n} = 0 \right\} = \left\{ \forall k \in \mathbb{N},\ \frac{|X_n|}{n} < \frac{1}{k} \ \forall \text{large } n \right\} = \bigcap_{k=1}^{\infty} \Omega_{1/k}$$

$$\therefore \mathbb{P}(\qquad) = 1.$$

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Back to the "realistic" lightbulb problem:

$$\{X_n\}_{n=1}^{\infty} \text{ iid } L^1, \quad \{Y_n\}_{n=1}^{\infty} \text{ iid } L^1$$

$$\mathbb{P}(X_n > 0) > 0, \quad \mathbb{P}(Y_n > 0) > 0.$$

$$S_n = X_1 + Y_1 + \cdots + X_n + Y_n, \quad N_t = \sup\{n \in \mathbb{N} : S_n \le t\}$$

$$\frac{N_t}{t} \xrightarrow[t\to\infty]{} \left(\mathbb{E}[X_1] + \mathbb{E}[Y_1]\right)^{-1} \text{ a.s.}$$

$T_t :=$ length of time in $[0,t]$ that a working lightbulb is installed.

$$= \sum_{n=1}^{N_t} X_n + X_{N_t+1} \wedge (t - S_{N_t}) \qquad \tilde{T}_t = \sum_{n=1}^{N_t} X_n$$

$$\frac{\tilde{T}_t}{t} \le \frac{T_t}{t} \le \frac{\tilde{T}_t + X_{N_t+1}}{t}$$

$$\frac{X_{N_t+1}}{t} = \underbrace{\frac{X_{N_t+1}}{N_t+1}}_{\to\, 0} \cdot \underbrace{\frac{N_t+1}{t}}_{\to\, \frac{1}{\mathbb{E}[X_1+Y_1]}} \quad \text{a.s.}$$

$$\therefore \lim_{t\to\infty} \frac{T_t}{t} = \lim_{t\to\infty} \frac{\tilde{T}_t}{t} = \lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N_t} X_n \cdot \frac{N_t}{N_t}$$

$$= \lim_{t\to\infty} \underbrace{\frac{N_t}{t}}_{\to\ \frac{1}{\mathbb{E}[X_1]+\mathbb{E}[Y_1]}} \cdot \underbrace{\frac{\sum_{n=1}^{N_t} X_n}{N_t}}_{\to\ \mathbb{E}[X_1]}$$

<u>Conclusion:</u>

$$\lim_{t\to\infty} \frac{T_t}{t} = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1]+\mathbb{E}[Y_1]} \quad //$$