We now have some (Kolmogorov) tools to prove a.s. convergence of a sum $\sum_{n=1}^{\infty} Y_n$, given information about $\mathrm{Var}(Y_n)$.

Not well-adapted to $\frac{1}{n}\sum_{j=1}^{n} X_j$; more adapted to $\sum_{n=1}^{\infty} \frac{X_n}{n}$.

Lemma: (Kronecker)

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$ (or any normed space) and let $\{b_k\}_{k=1}^{\infty} \subset (0,\infty)$ be an increasing sequence $b_k \uparrow \infty$.

If $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{b_k}$ exists in $\mathbb{R}$, then $\lim\limits_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} x_k = 0$.

Pf. Let $y_k := \frac{x_k}{b_k}$, $S_n := \sum_{k=1}^{n} y_k$ $(S_0 := 0)$, $\lim_{n\to\infty} S_n =: s$

Then $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} b_k y_k$

$$\therefore \quad \frac{1}{b_n}\sum_{k=1}^{n} x_k = S_n - \frac{1}{b_n}\sum_{k=1}^{n-1}(b_{k+1}-b_k)S_k$$

$$= S_n - \frac{1}{b_n}\sum_{k=1}^{n-1}(b_{k+1}-b_k)s + R_n$$

Theorem: (Kolmogorov's Strong Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. L^1 random variables with $E[X_n] = \alpha$.
Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \to \alpha \quad \text{a.s.}$$

We already showed that it suffices to show $\frac{S_n'}{n} \to \mu$ a.s., where

$$S_n' := \sum_{j=1}^{n} X_j' , \qquad X_j' := X_j \mathbb{1}_{|X_j| \le j}$$

We'll now apply:

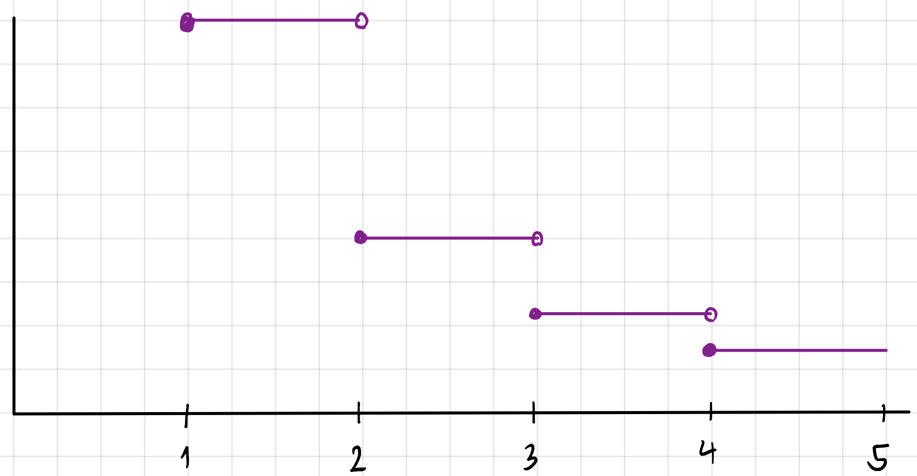
Theorem: (Kolmogorov's Convergence Criterion)

Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables.
If $\sum_{n\ge1}^{\infty} \operatorname{Var}(Y_n) < \infty$, then $\sum_{n\ge1}^{\infty} \mathring{Y}_n$ converges a.s.

$$\sum_{n=1}^{\infty} \mathrm{Var}\left(\frac{X_n'}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{Var}(X_n') \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}[|X_n'|^2]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}\left[|X_n|^2 \mathbb{1}_{|X_n| \leq n}\right]$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{1}_{x \leq n} = \sum_{n \geq x} \frac{1}{n^2} \leq \int_x^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^2} \mathbb{1}_{n \leq t < n+1} \right) dt \quad \text{for } x > 1 .$$



$\therefore$ By Kolmogorov's Convergence Criterion,

$\sum_{n=1}^{\infty} \left(\frac{X_n'}{n} - \mathbb{E}\left[\frac{X_n'}{n}\right] \right)$ converges a.s.

$$\sum_{n=1}^{\infty}\left(\frac{X_n'}{n}-\mathbb{E}\left[\frac{X_n'}{n}\right]\right)=\sum_{n=0}^{\infty}\frac{1}{n}\left(X_n'-\mathbb{E}[X_n']\right)$$ converges a.s.

$\therefore$ By Kronecker's Lemma,

$$\Rightarrow \quad \mathring{S}_n' := \frac{1}{n}\sum_{k=1}^{n}\left(X_k'-\mathbb{E}[X_k']\right)\to 0 \quad \text{a.s.}$$

$$\|$$

$$\frac{1}{n}\sum_{k=1}^{n}X_k' - \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[X_k \mathbb{1}_{|X_k|\le k}\right]$$

For each k, let $\alpha_k = \mathbb{E}\left[X_1 \mathbb{1}_{|X_1|\le k}\right]$.

Rates of Convergence

How fast does $\frac{S_n}{n}$ converge?
I.e. if the common mean is α,

$$\left| \frac{S_n}{n} - \alpha \right| = o(?)$$

That is: what is the fastest growing $a_n \uparrow \infty$ s.t.

$$\limsup_{n\to\infty} a_n \cdot \left| \frac{S_n}{n} - \alpha \right| < \infty \ ?$$

Theorem: (Marcinkiewicz, Zygmund)
Suppose $\{X_n\}_{n=1}^{\infty}$ are iid in L^p for some $p \in (1,2)$.
Then

$$\frac{S_n - n\alpha}{n^{1/p}} \to 0 \quad a.s.$$

The proof is nearly identical to the one we just went through.

- Use $X_n' = X_n \mathbb{1}_{|X_n| \le n^{1/p}}$. $\sum_{n \ge x} n^{-2/p} \le \frac{p}{2-p} (x-1)^{\frac{p-2}{p}}$

Theorem: [26.15] (L^2 - SLLN)

Let $\{X_n\}_{n=1}^{\infty}$ be independent L^2 random variables, with common mean $E[X_n] = \alpha$ and variance $Var[X_n] \leq s^2$.

Let $S_n = X_1 + \cdots + X_n$, and let $b_n > 0$ s.t. $\sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$.

Then $$\frac{S_n - n\alpha}{b_n} \to 0 \quad \text{a.s. and in } L^2.$$

Pf. $\sum_{n=1}^{\infty} Var\left(\frac{X_n}{b_n}\right)$

$\therefore$ By Kolmogorov's Convergence Criterion, $\sum_{n=1}^{\infty} \frac{\mathring{X}_n}{b_n}$

$\therefore$ By Kronecker's Lemma, $\frac{1}{b_n} \sum_{k=1}^{n} \mathring{X}_n$

For L^2 Convergence: $\left\| \frac{S_n - n\alpha}{b_n} \right\|_{L^2}^2 = E\left[\left(\frac{S_n - n\alpha}{b_n} \right)^2 \right]$

So, $\frac{n}{b_n} \cdot \left| \frac{S_n}{n} - \alpha \right| \to 0$ a.s.

The Law of the Iterated Logarithm (Khinchin)

If $\{X_n\}_{n=1}^{\infty}$ are independent L^2 random variables common mean $E[X_n] = \alpha$ and common variance $Var[X_n] = s^2$, and $S_n = X_1 + \cdots + X_n$, then

$$\limsup_{n \to \infty} \frac{S_n - n\alpha}{\sqrt{2s^2 n \log\log n}} = 1 \quad a.s.$$