We now have some (Kolmogorov) tools to prove a.s. convergence of a sum $\sum_{n=1}^{\infty} Y_n$, given information about $\mathrm{Var}(Y_n)$.

Not well-adapted to $\frac{1}{n}\sum_{j=1}^{n} X_j$; more adapted to $\sum_{n=1}^{\infty} \frac{X_n}{n}$.

Lemma: (Kronecker)

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$ (or any normed space) and let $\{b_k\}_{k=1}^{\infty} \subset (0,\infty)$ be an increasing sequence $b_k \uparrow \infty$.

If $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{b_k}$ exists in $\mathbb{R}$, then $\lim_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n} x_k = 0$.

Pf. Let $y_k := \frac{x_k}{b_k}$, $S_n := \sum_{k=1}^{n} y_k$ $(S_0 := 0)$, $\lim_{n\to\infty} S_n =: s$

$$y_k = S_k - S_{k-1}$$

Then
$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} b_k y_k = \sum_{k=1}^{n} b_k (S_k - S_{k-1})$$
$$= \sum_{k=1}^{n} b_k S_k - \sum_{k=0}^{n-1} b_{k+1} S_k = b_n S_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) S_k$$

$$\therefore \quad \frac{1}{b_n} \sum_{k=1}^{n} x_k = S_n - \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) S_k$$
$$= S_n - \frac{1}{b_n} \underbrace{\sum_{k=1}^{n-1} (b_{k+1} - b_k)}_{b_n - b_1} s + R_n$$

$$\hookrightarrow R_n = \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k)(s - S_k)$$

$$s - S_k = \sum_{j \geq k+1}^{\infty} y_j$$

$$|s - S_k| \to 0 \text{ as } k \to \infty$$

$$S_n - \left(1 - \frac{b_1}{b_n}\right) s + R_n$$

$$|R_n| \leq \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) |s - S_k|$$
$$\leq \frac{M}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) = M\left(1 - \frac{b_1}{b_n}\right) \leq M.$$

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$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{1}{b_n} \sum_{k=N}^{n} (b_{k+1} - b_k) |s - S_k|$$
$$\leq \lim_{n\to\infty} \sup_{k \geq N} |s - S_k| \cdot \underbrace{\frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k)}_{\left(1 - \frac{b_1}{b_n}\right)}$$
$$\therefore \ = 0.$$

Theorem: (Kolmogorov's Strong Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. L^1 random variables with $E[X_n] = \alpha$.
Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \to \alpha \quad \text{a.s.}$$

We already showed that it suffices to show $\frac{S_n'}{n} \to \mu$ a.s., where

$$S_n' := \sum_{j=1}^{n} X_j', \qquad X_j' := X_j \mathbb{1}_{|X_j| \le j}$$

We'll now apply:

Theorem: (Kolmogorov's Convergence Criterion)

Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables.
If $\sum_{n \ge 1}^{\infty} Var(Y_n) < \infty$, then $\sum_{n \ge 1}^{\infty} Y_n$ converges a.s.

$$Y_n = \frac{X_n'}{n}$$

$$\sum_{n=1}^{\infty} Var\left(\frac{X_n'}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} Var(X_n') \le \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}[|X_n'|^2]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}[|X_1|^2 \mathbb{1}_{|X_1| \le n}]$$

$$= \mathbb{E}\left[|X_1|^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{1}_{|X_1| \le n}\right]$$

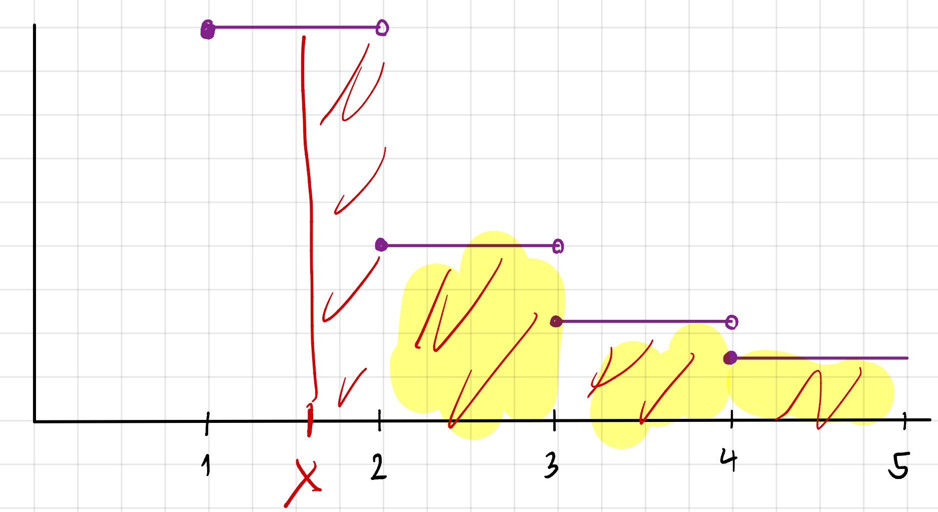
for fixed t $= \dfrac{1}{\lfloor t \rfloor^2} \le \dfrac{1}{(t-1)^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{1}_{x \le n} = \sum_{n \ge x} \frac{1}{n^2} \le \int_x^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^2} \mathbb{1}_{n \le t < n+1} \right) dt \quad \text{for } x > 1.$$

$$\le \int_x^{\infty} \frac{1}{(t-1)^2} dt = \frac{1}{x-1} \le \frac{2}{x}.$$

$x \le 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \doteq 1.645 \le 2$

$$\le \min\left(2, \frac{2}{x}\right)$$



$$\le \mathbb{E}\left[|X_1|^2 \min\left(2, \frac{2}{|X_1|}\right)\right] \le 2\,\mathbb{E}[|X_1|] < \infty.$$

$\therefore$ By Kolmogorov's Convergence Criterion,

$$\sum_{n=1}^{\infty} \left(\frac{X_n'}{n} - \mathbb{E}\left[\frac{X_n'}{n}\right] \right)$$ converges a.s.

$$\sum_{n=1}^{\infty}\left(\frac{X_n'}{n}-\mathbb{E}\left[\frac{X_n'}{n}\right]\right)=\sum_{n=0}^{\infty}\frac{1}{n}\left(X_n'-\mathbb{E}[X_n']\right) \text{ converges a.s.}$$

$\therefore$ By Kronecker's Lemma,

$$\Rightarrow \quad \mathring{S}_n' := \frac{1}{n}\sum_{k=1}^{n}\left(X_k'-\mathbb{E}[X_k']\right)\to 0 \quad \text{a.s.}$$

$$\| $$

$$\frac{1}{n}\sum_{k=1}^{n}X_k' - \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[X_k \mathbb{1}_{|X_k|\le k}\right]$$

$$\frac{S_n'}{n} \xrightarrow[\text{WTS}]{} \alpha \text{ a.s.}$$

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[X_1 \mathbb{1}_{|X_1|\le k}\right]$$

$\Leftarrow \ |X_1|\in L^1$ DCT.

$$\mathbb{E}\left[X_1 \mathbb{1}_{|X_1|\le k}\right]\to \mathbb{E}[X_1]=\alpha.$$

For each k, let $\alpha_k=\mathbb{E}\left[X_1 \mathbb{1}_{|X_1|\le k}\right] \to \alpha$.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\alpha_k=\alpha.$$

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Rates of Convergence

How fast does $\frac{S_n}{n}$ converge?
I.e. if the common mean is α,

$$\left| \frac{S_n}{n} - \alpha \right| = o(?)$$

That is: what is the fastest growing $a_n \uparrow \infty$ s.t.

$$\limsup_{n \to \infty} a_n \cdot \left| \frac{S_n}{n} - \alpha \right| < \infty \ ?$$

Theorem: (Marcinkiewicz, Zygmund)
Suppose $\{X_n\}_{n=1}^{\infty}$ are iid in L^p for some $p \in (1,2)$.
Then $n^{1-\frac{1}{p}}\left(\frac{S_n}{n} - \alpha\right) = \frac{S_n - n\alpha}{n^{1/p}} \longrightarrow 0$ a.s. $\quad \left|\frac{S_n}{n} - \alpha\right| = o\left(\frac{1}{n^{1-\frac{1}{p}}}\right)$

The proof is nearly identical to the one we just went through.

- Use $X_n' = X_n \mathbb{1}_{|X_n| \leq n^{1/p}}$
- $\sum_{n \geq x} n^{-2/p} \leq \frac{p}{2-p} (x-1)^{\frac{p-2}{p}}$

Theorem: [26.15] (L^2-SLLN)

Let $\{X_n\}_{n=1}^{\infty}$ be independent L^2 random variables, with common mean $E[X_n]=\alpha$ and variance $Var[X_n] \le s^2$.

Let $S_n = X_1 + \cdots + X_n$, and let $b_n > 0$ s.t. $\sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$.

Then $\frac{S_n - n\alpha}{b_n} \to 0$ a.s. and in L^2.

Pf. $\sum_{n=1}^{\infty} Var\left(\frac{X_n}{b_n}\right) \le \sum_{n=1}^{\infty} \frac{1}{b_n^2} Var(X_n) \le s^2 \sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$.

$\therefore$ By Kolmogorov's Convergence Criterion, $\sum_{n=1}^{\infty} \frac{\mathring{X}_n}{b_n}$ exists in $\mathbb{R}$ a.s.

$\therefore$ By Kronecker's Lemma, $\frac{1}{b_n} \sum_{k=1}^{n} \mathring{X}_n \longrightarrow 0$ a.s.

For L^2 Convergence: $\left\| \frac{S_n - n\alpha}{b_n} \right\|_{L^2}^2 = E\left[\left(\frac{S_n - n\alpha}{b_n}\right)^2\right] = \frac{1}{b_n^2} Var(S_n) = \frac{1}{b_n^2} \sum_{k=1}^{n} Var(X_n)$

$\le s^2 \cdot \frac{1}{b_n^2} \sum_{k=1}^{n} 1$.

E.g. $b_n = n^p$ $(p > \frac{1}{2})$

So, $\frac{n}{b_n} \cdot \left| \frac{S_n}{n} - \alpha \right| \to 0$ a.s.

$b_n = \sqrt{n}\,(\log n)^{\frac{1}{2}+\varepsilon}$ $\varepsilon > 0$. $\to 0$ by Kronecker

The Law of the Iterated Logarithm (Khinchin)

If $\{X_n\}_{n=1}^{\infty}$ are independent L^2 random variables common mean $E[X_n] = \alpha$ and common variance $Var[X_n] = s^2$, and $S_n = X_1 + \cdots + X_n$, then

$$\limsup_{n \to \infty} \frac{S_n - n\alpha}{\sqrt{2 s^2 n \log \log n}} = 1 \quad a.s.$$

$$\frac{S_n - n\alpha}{n} = O\left(\sqrt{\frac{\log\log n}{n}}\right)$$