Let $\{Y_n\}_{n=1}^{\infty}$ be uncorrelated random variables in L^2.

I.e. $\mathrm{Cov}(Y_n, Y_m) = 0$ for $n \neq m$

Prop: If $\{Y_n\}_{n=1}^{\infty}$ are uncorrelated, and $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$, then $\sum_{n=1}^{\infty} (Y_n - \mathbb{E}[Y_n])$ converges in L^2.

Pf. $\mathrm{Var}(Y_n) = \mathrm{Cov}(Y_n, Y_n)$

We would like to upgrade this convergence from L^2 to a.s. There's no reason for that to be true in general (orthogonal sums in L^2 usually fail to converge a.s.), unless we also upgrade the orthogonality to super-orthogonality – i.e. independence.

Theorem: (Kolmogorov's Convergence Criterion)
Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables.
If $\sum_{n=1}^{\infty} Var(Y_n) < \infty$, then $\sum_{n=1}^{\infty} (Y_n - \mathbb{E}[Y_n])$ converges a.s.
In particular, if in addition $\sum_{n=1}^{\infty} \mathbb{E}[Y_n] < \infty$, then $\sum_{n=1}^{\infty} Y_n$ converges a.s. and in L^2.

Maximal Inequalities

Let $\{Y_n\}_{n=1}^{\infty}$ be independent rv's, with $E[Y_n]=0$

Set $S_n = Y_1 + \cdots + Y_n$. If $Y_n \in L^2$, then Markov tells us

$$P(|S_n| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E[S_n^2]$$

But what can we say about

$$S_n^* := \max_{1 \leq j \leq n} |S_j|$$

Turns out: the Markov conclusion still applies.

Theorem: (Kolmogorov's Maximal Inequality)

with Y_n, S_n as above,

$$P(S_n^* \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E[S_n^2 \qquad \forall \varepsilon > 0.$$

Pf. Fix $\varepsilon > 0$, and set $\tau := \inf\{ j \in \mathbb{N} : |S_j| \geq \varepsilon \}$

τ is a random variable. Note that

$\{\tau = j\}$

Now, $\{S_n^* \geq \varepsilon\}$

Notation: $\mathbb{E}[X : A] := \mathbb{E}[X \mathbb{1}_A]$

$\therefore \mathbb{E}[S_n^2 : S_n^* \geq \varepsilon] = \mathbb{E}[S_n^2 : \quad] = \sum_{j=1}^{n} \mathbb{E}[S_n^2 : \quad]$

Now a trick:

$$S_n^2 = (S_j + S_n - S_j)^2 = S_j^2 + (S_n - S_j)^2 + 2 \underbrace{S_j}_{Y_1 + \cdots + Y_j} \underbrace{(S_n - S_j)}_{Y_{j+1} + \cdots + Y_n}$$

$$\mathbb{E}[S_n^2 : S_n^* \geq \varepsilon] = \sum_{j=1}^{n} \mathbb{E}\left[\left(S_j^2 + (S_n - S_j)^2 + 2S_j(S_n - S_j)\right)\mathbb{1}_{\{\tau = j\}}\right]$$

$$\downarrow$$

$$\mathbb{E}\left[(S_n - S_j) \cdot S_j \mathbb{1}_{\{\tau = j\}}\right]$$

$$= \sum_{j=1}^{n} \left(\mathbb{E}[S_j^2 : \tau = j] + \mathbb{E}[(S_n - S_j)^2 : \tau = j]\right)$$

$$\geq \sum_{j=1}^{n} \mathbb{E}[S_j^2 : \tau = j]$$

Theorem: (Kolmogorov's Convergence Criterion)

Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables.

If $\sum_{n\geq 1}^{\infty} Var(Y_n) < \infty$, then $\sum_{n\geq 1}^{\infty} \mathring{Y}_n$ converges a.s.

Pf. Let $S_n = \sum_{j=1}^{n} \mathring{Y}_j$. For $m<n$, $S_n - S_m = \mathring{Y}_{m+1} + \cdots + \mathring{Y}_n$.

Apply Kolmogorov's Maximal inequality:

$$\mathbb{P}\left(\max_{m<j\leq n} |S_j - S_m| \geq \frac{\varepsilon}{2}\right) \leq \frac{1}{(\varepsilon/2)^2} E\left[(S_n - S_m)^2\right] = \frac{4}{\varepsilon^2} \sum_{j=m+1}^{n} E[\mathring{Y}_j^2]$$

Now let $n \to \infty$; we have

$$\mathbb{P}\left(\sup_{j\geq m} |S_j - S_m| \geq \frac{\varepsilon}{2}\right) \leq \frac{4}{\varepsilon^2} \sum_{j=m+1}^{\infty} Var(Y_j)$$

Further, $\sup_{j,k\geq m} |S_j - S_k| = \sup_{j,k\geq m} |S_j - S_m + S_m - S_k| \leq \sup_{j\geq m} |S_j - S_m| + \sup_{k\geq m} |S_k - S_m|$

$$= 2 \sup_{j\geq m} |S_j - S_m|$$

$\therefore$ We have proven that, $\forall \varepsilon > 0$,

$$\mathbb{P}\Big(\sup_{j,k \geq m} |S_j - S_k| \geq \varepsilon \Big) \longrightarrow 0 \quad \text{as} \quad m \to \infty$$

I.e. the random variables $\delta_m := \sup_{j,k \geq m} |S_j - S_k| \longrightarrow_p 0$.

But $\delta_m \downarrow$ and $\delta_m \geq 0$. $\therefore$ $\delta := \lim_{m \to \infty} \delta_m$ exists (surely).

Eg. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d Rademacher rv's: $\mathbb{P}(X_n = \pm 1) = \frac{1}{2}$.

Does the series $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converge? $\Big[\pm \sum_{n=1}^{\infty} \frac{1}{n}$ diverges; $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges $\Big]$