Let $\{Y_n\}_{n=1}^{\infty}$ be uncorrelated random variables in L^2.

I.e. $\quad Cov(Y_n, Y_m) = 0 \quad$ for $\quad n \neq m$

$$\overset{\parallel}{E[\mathring{Y}_n \mathring{Y}_m]} = \langle \mathring{Y}_n, \mathring{Y}_m \rangle_{L^2} \qquad \mathring{Y}_n = Y_n - E[Y_n]$$

i.e. $\{\mathring{Y}_n\}$ is an orthogonal seq. in L^2.

Prop: If $\{Y_n\}_{n=1}^{\infty}$ are uncorrelated, and $\sum_{n=1}^{\infty} Var(Y_n) < \infty$, then $\sum_{n=1}^{\infty}(Y_n - E[Y_n])$ converges in L^2.

Pf. $Var(Y_n) = Cov(Y_n, Y_n) = E[\mathring{Y}_n^2] = \|\mathring{Y}_n\|_{L^2}^2$

$$\mathring{S}_n = \sum_{j=1}^{n} \mathring{Y}_j \qquad \|\mathring{S}_n - \mathring{S}_m\|_{L^2}^2 = \Big\| \sum_{j=m+1}^{n} \mathring{Y}_j \Big\|_{L^2}^2 = \sum_{j=m+1}^{n} \|\mathring{Y}_j\|_{L^2}^2 \to 0$$

as $n, m \to \infty$,

$\therefore \mathring{S}_n = \sum_{j=1}^{n}(Y_j - E[Y_j])$ is Cauchy in L^2,

$\therefore$ convergent.

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We would like to upgrade this convergence from L^2 to a.s. There's no reason for that to be true in general (orthogonal sums in L^2 usually fail to converge a.s.), unless we also upgrade the orthogonality to super-orthogonality – i.e. independence.

Theorem: (Kolmogorov's Convergence Criterion)
Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables.
If $\sum_{n=1}^{\infty} Var(Y_n) < \infty$, then $\sum_{n=1}^{\infty} (Y_n - \mathbb{E}[Y_n])$ converges a.s.
In particular, if in addition $\sum_{n=1}^{\infty} \mathbb{E}[Y_n] < \infty$, then $\sum_{n=1}^{\infty} Y_n$ converges a.s. and in L^2.

$\sum_n Y_n = \sum_n \overset{\circ}{Y}_n + \sum_n \mathbb{E}[Y_n]$

Maximal Inequalities

Let $\{Y_n\}_{n=1}^{\infty}$ be independent rv's, with $E[Y_n]=0$

Set $S_n = Y_1 + \cdots + Y_n$. If $Y_n \in L^2$, then Markov tells us

$$P(|S_n| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E[S_n^2] = \frac{1}{\varepsilon^2} \sum_{j=1}^{n} E[Y_j^2]$$

But what can we say about

$$S_n^* := \max_{1 \leq j \leq n} |S_j|$$

Turns out: the Markov conclusion still applies.

Theorem: (Kolmogorov's Maximal Inequality)

with Y_n, S_n as above,

$$P(S_n^* \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E[S_n^2 \mathbb{1}_{S_n^* \geq \varepsilon}] \quad \forall \varepsilon > 0.$$

$$\leq \frac{1}{\varepsilon^2} E[S_n^2] = \frac{1}{\varepsilon^2} \sum_{j=1}^{n} E[Y_j^2].$$

Pf. Fix $\varepsilon > 0$, and set $\tau := \inf\{ j \in \mathbb{N} : |S_j| \geq \varepsilon \}$

$(\inf \emptyset := \infty)$

τ is a random variable. Note that

$$\{\tau = j\} = \{ |S_1| < \varepsilon, |S_2| < \varepsilon, \dots, |S_{j-1}| < \varepsilon, |S_j| \geq \varepsilon \}$$

$$\{\tau = j\} \in \sigma(Y_1, \dots, Y_j), \quad |S_1| = |Y_1|, \quad |S_2| = |Y_1 + Y_2|, \quad |S_{j-1}| = |Y_1 + \dots + Y_{j-1}|, \quad |S_j| = |Y_1 + \dots + Y_j|$$

Now, $\{S_n^* \geq \varepsilon\} = \{ \max_{1 \leq j \leq n} |S_j| \geq \varepsilon \} = \{ \exists j \in [n] : |S_j| \geq \varepsilon \} = \{\tau \leq n\}$

Notation: $\mathbb{E}[X : A] := \mathbb{E}[X \mathbb{1}_A]$

$$\therefore \mathbb{E}[S_n^2 : S_n^* \geq \varepsilon] = \mathbb{E}[S_n^2 : \tau \leq n] = \sum_{j=1}^n \mathbb{E}[S_n^2 : \tau = j]$$

$$\mathbb{1}_{\{\tau \leq n\}} = \sum_{j=1}^n \mathbb{1}_{\{\tau = j\}}$$

Now a trick:

$$S_n^2 = (S_j + S_n - S_j)^2 = S_j^2 + (S_n - S_j)^2 + 2 S_j (S_n - S_j)$$

$$S_j = Y_1 + \dots + Y_j, \qquad S_n - S_j = Y_{j+1} + \dots + Y_n$$

$$\mathbb{E}[S_n^2 : S_n^* \geq \varepsilon] = \sum_{j=1}^{n} \mathbb{E}\left[\left(S_j^2 + (S_n - S_j)^2 + \cancel{2S_j(S_n - S_j)}\right)\mathbb{1}_{\{\tau = j\}}\right]$$

$$\mathbb{E}[(S_n - S_j) \cdot S_j \mathbb{1}_{\{\tau = j\}}] = \mathbb{E}[\cancel{S_n - S_j}]\,\mathbb{E}[S_j : \tau = j]$$

$\sigma(Y_{j+1}, \dots, Y_n)$ $\qquad$ $\sigma(Y_1, \dots, Y_j)$ $\qquad$ 0

$$= \sum_{j=1}^{n} \left(\mathbb{E}[S_j^2 : \tau = j] + \mathbb{E}[(S_n - S_j)^2 : \tau = j]\right)$$

$$\geq \sum_{j=1}^{n} \underbrace{\mathbb{E}[S_j^2 : \tau = j]}_{\mathbb{E}[S_j^2 \mathbb{1}_{\{\tau = j\}}]}$$

$$\{\tau = j\} \subseteq \{|S_j| \geq \varepsilon\}$$

$$S_j^2 \mathbb{1}_{\{\tau = j\}} \geq \varepsilon^2 \mathbb{1}_{\{\tau = j\}}$$

$$\geq \sum_{j=1}^{n} \mathbb{E}[\varepsilon^2 \mathbb{1}_{\{\tau = j\}}] = \varepsilon^2 \sum_{j=1}^{n} \mathbb{P}(\tau = j) = \varepsilon^2 \mathbb{P}(\tau \leq n)$$

$$= \varepsilon^2 \mathbb{P}(S_n^* \geq \varepsilon)$$

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Theorem: (Kolmogorov's Convergence Criterion)

Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables.

If $\sum_{n\geq 1}^{\infty} Var(Y_n) < \infty$, then $\sum_{n\geq 1}^{\infty} \overset{\circ}{Y}_n$ converges a.s.

Pf. Let $S_n = \sum_{j=1}^{n} \overset{\circ}{Y}_j$. For $m < n$, $S_n - S_m = \overset{\circ}{Y}_{m+1} + \cdots + \overset{\circ}{Y}_n$.

Apply Kolmogorov's Maximal inequality:

$$\mathbb{P}\left(\max_{m<j\leq n} |S_j - S_m| \geq \frac{\varepsilon}{2}\right) \leq \frac{1}{(\varepsilon/2)^2} \mathbb{E}\left[(S_n - S_m)^2\right] = \frac{4}{\varepsilon^2} \sum_{j=m+1}^{n} \mathbb{E}[\overset{\circ}{Y}_j{}^2] = \frac{4}{\varepsilon^2} \sum_{j=m+1}^{n} Var(Y_j)$$

Now let $n \to \infty$; we have

$$\mathbb{P}\left(\sup_{j\geq m} |S_j - S_m| \geq \frac{\varepsilon}{2}\right) \leq \frac{4}{\varepsilon^2} \sum_{j=m+1}^{\infty} Var(Y_j) \longrightarrow 0 \text{ as } m \to \infty,$$

Further, $\sup_{j,k\geq m} |S_j - S_k| = \sup_{j,k\geq m} |S_j - S_m + S_m - S_k| \leq \sup_{j\geq m} |S_j - S_m| + \sup_{k\geq m} |S_k - S_m|$

$$= 2 \sup_{j\geq m} |S_j - S_m|$$

$$\left\{\sup_{j,k\geq m} |S_j - S_k| \geq \varepsilon\right\} \subseteq \left\{2 \sup_{j\geq m} |S_j - S_m| \geq \varepsilon\right\}$$

$$\mathbb{P}\Big(\qquad\qquad\Big) \leq \mathbb{P}\Big(\qquad\qquad\Big) \longrightarrow 0 \text{ as } m \to \infty.$$

$\therefore$ We have proven that, $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\sup_{j,k \geq m} |S_j - S_k| \geq \varepsilon \right) \longrightarrow 0 \quad \text{as } m \to \infty$$

I.e. the random variables $\delta_m := \sup_{j,k \geq m} |S_j - S_k| \longrightarrow_p 0$.

But $\delta_m \downarrow$ and $\delta_m \geq 0$. $\therefore$ $\delta := \lim_{m\to\infty} \delta_m$ exists (surely). $\Big\}$ $\delta = 0$.

$$\therefore \sup_{j,k \geq m} |S_j - S_k| \longrightarrow 0 \quad \text{a.s. as } m \to \infty$$

I.e. $\{S_j\}_{j=1}^{\infty}$ is a.s. Cauchy.

$\underset{\displaystyle \sum_{k=1}^{j} \mathring{Y}_k}{\parallel}$ $\therefore$ a.s. convergent.

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E.g. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d Rademacher rv's: $\mathbb{P}(X_n = \pm 1) = \frac{1}{2}$.

Does the series $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converge? $\left[\pm \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges; } \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges} \right]$

$Y_n = \frac{X_n}{n}$ indep., L^2, $\mathring{Y}_n = Y_n$ $\quad \mathrm{Var}(Y_n) = \mathbb{E}[\mathring{Y}_n^2] = \frac{1}{n^2}\mathbb{E}[X_n^2] = \frac{1}{n^2}$

$\therefore \sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$.

$\therefore \sum_{n=1}^{\infty} \mathring{Y}_n = \sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s.