The Law of Large Numbers: Revisited

Recall the weak Law of Large Numbers: (Lec 12.2)

Let $\{X_n\}_{n\geq 1}^{\infty}$ be uncorrelated L^2 random variables

and suppose that $\mathbb{E}[X_n] = \alpha \ \forall n$, $\mathbb{E}[X_n^2] = s^2 \ \forall n$.

Set $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \to_p \alpha .$$

The proof was a simple application of Chebyshev's inequality.

There are at least two ways we could improve the result:

1. weaken the hypothesis that $X_n \in L^2$.
2. Strengthen the convergence from $\to_p$.

Cut-Offs

Let X be any random variable.

Let $M < \infty$. Then

$$\underset{\sigma(X)\text{ - measurable}}{\underbrace{X \mathbb{1}_{|X| \leq M}}} \quad \text{is bounded}$$

Only trouble: $\mathrm{Cov}(X, Y) = 0 \not\Rightarrow \mathrm{Cov}(X \mathbb{1}_{|X| \leq M}, Y \mathbb{1}_{|Y| \leq M}) = 0$.

Solution: trade up. Replace the weak uncorrelated assumption with a stronger (and natural) independence assumption.

$\therefore$ We can replace X_n with $X_n \mathbb{1}_{|X_n| \leq M_n}$
at the expense of assuming full independence.
The idea will then be to "remove" the cut-off.

This approach does work (with some work!)
and we're going to follow it to prove:

Theorem: (Kolmogorov's Strong Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. L^1 random variables with $E[X_n] = \alpha$.
Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \to \alpha \quad \text{a.s.}$$

Cor: If $X_n \notin L^1$ but $X_n^- \in L^1$ (so "$E[X_n] = +\infty$"), then $\frac{S_n}{n} \to +\infty$ a.s.

Pf.

Tail Equivalence

Def: Two sequences $\{X_n\}_{n=1}^{\infty}$, $\{X_n'\}_{n=1}^{\infty}$ on a common probability space are called **tail equivalent** if

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq X_n') < \infty$$

By the Borel-Cantelli Lemma (I), setting $A_n = \{X_n \neq X_n'\}$, we have $\mathbb{P}(A_n \text{ i.o.}) = 0$.

I.e. $\exists$ null set N s.t. $\forall \omega \in N^c$,

Cor: If $\{X_n\}_{n=1}^{\infty}$, $\{X_n'\}_{n=1}^{\infty}$ are tail equivalent, and $b_n \uparrow \infty$, if $\exists$ r.v. X s.t.

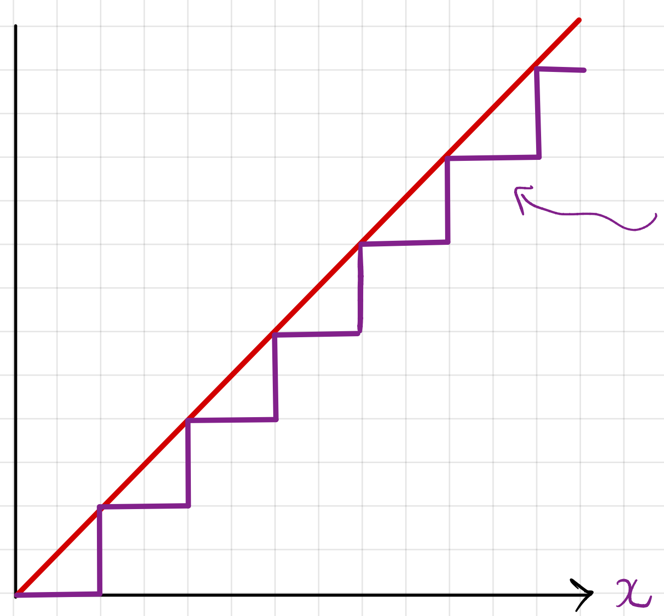
$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} X_j' = X \quad \text{a.s.}, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} X_j = X \quad \text{a.s.}$$

We'd like to find a sequence of cut-offs $X_n' = X_n \mathbb{1}_{|X_n| \le M_n}$ so that $\{X_n\}_{n=1}^{\infty}$, $\{X_n'\}_{n=1}^{\infty}$ are tail equivalent. To that end, we have:

Lemma: If $X \in L^1$ and $\varepsilon > 0$, then

$$\sum_{n=1}^{\infty} \mathbb{P}(|X| \ge n\varepsilon) \le \frac{1}{\varepsilon} \mathbb{E}[|X|].$$

Pf.



$$x \mapsto \sum_{n=1}^{\infty} \mathbb{1}_{[n,\infty)}(x)$$

Cor: If $\{X_n\}_{n=1}^{\infty}$ are (i)id and L^1, they are tail equivalent to $X_n' = X_n \mathbb{1}_{|X_n| \le n}$.

Pf.

Thus, in order to prove the SLLN, it suffices to prove:

If $\{X_n\}_{n\geq 1}^{\infty}$ is an iid sequence of L^1 random variables with $\mathbb{E}[X_n]=\alpha$, and

$$S_n' = \sum_{k=1}^{n} X_k \mathbb{1}_{|X_k|\leq k},$$

then $\frac{S_n'}{n} \to \alpha$ a.s.

Advantages:

Disadvantages: