The Law of Large Numbers: Revisited

Recall the Weak Law of Large Numbers: (Lec 12.2)

Let $\{X_n\}_{n\geq 1}^{\infty}$ be uncorrelated L^2 random variables

$$\mathrm{Cov}(X_n, X_m) = 0. \quad \forall n \neq m$$

and suppose that $\mathbb{E}[X_n] = \alpha \ \ \forall n$, $\mathbb{E}[X_n^2] = s^2 \ \ \forall n$.

Set $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \to_p \alpha. \quad \text{In fact } \mathbb{P}\left(\left|\frac{S_n}{n} - \alpha\right| > \varepsilon\right) \leq \frac{s^2}{\varepsilon^2} \cdot \frac{1}{n}.$$

The proof was a simple application of Chebyshev's inequality.

There are at least two ways we could improve the result:

1. weaken the hypothesis that $X_n \in L^2$. $X_n \in L^1$ should suffice.
2. Strengthen the convergence from $\to_p$. a.s.

Cutoff argument

Cut-Offs

Let X be any random variable.
Let $M < \infty$. Then

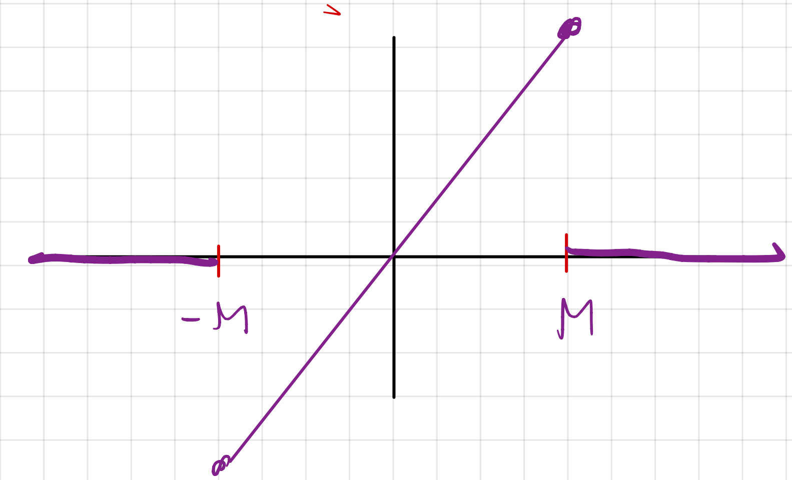
$X \mathbb{1}_{|X| \le M}$ is bounded hence in L^p $\forall p \ge 1$.

$E[|X \mathbb{1}_{|X|\le M}|^p] \le M^p < \infty$.

$\uparrow$

$\sigma(X)$-measurable

$X \mathbb{1}_{|X|\le M} = f_M(X)$ $\quad$ $f_M(x) = x \mathbb{1}_{|x| \le M}$
Borel fn.



Only trouble: $Cov(X, Y) = 0 \not\Rightarrow Cov(X\mathbb{1}_{|X|\le M}, Y\mathbb{1}_{|Y|\le M}) = 0$.

Solution: trade up. Replace the weak uncorrelated assumption with a stronger (and natural) independence assumption.

$\{X_n\}_{n=1}^{\infty}$ indep. $\Longleftrightarrow$ $\{\sigma(X_n)\}_{n=1}^{\infty}$ indep. $\Rightarrow$ $\{X_n \mathbb{1}_{|X_n| \le M_n}\}$ indep.

$\therefore$ We can replace X_n with $X_n \mathbb{1}_{|X_n| \le M_n}$ at the expense of assuming full independence.

The idea will then be to "remove" the cut-off.

This approach does work (with some work!) and we're going to follow it to prove:

Theorem: (Kolmogorov's Strong Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. L^1 random variables with $\mathbb{E}[X_n] = \alpha$. Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \to \alpha \quad \text{a.s.}$$

Cor: If $X_n \notin L^1$ but $X_n^- \in L^1$ (so "$\mathbb{E}[X_n] = +\infty$"), then $\frac{S_n}{n} \to +\infty$ a.s.

Pf. Fix $M < \infty$, let $X_n^M = X_n \wedge M$. $\therefore$ by SLLN, $\frac{S_n^M}{n} = \frac{X_1^M + \cdots X_n^M}{n} \to \mathbb{E}[X_1^M]$ a.s.

$$\therefore \frac{S_n}{n} \ge \frac{S_n^M}{n} \quad \therefore \liminf_{n\to\infty} \frac{S_n}{n} \ge \liminf_{n\to\infty} \frac{S_n^M}{n} = \mathbb{E}[X_1 \wedge M] = \mathbb{E}[\underbrace{(X_1 \wedge M)_+}_{X_1^+ \wedge M}] - \mathbb{E}[\underbrace{(X_1 \wedge M)^-}_{\le X^-}]$$

$\le X_n$. $\nearrow \infty$.

Tail Equivalence

Def: Two sequences $\{X_n\}_{n=1}^{\infty}$, $\{X_n'\}_{n=1}^{\infty}$ on a common probability space are called tail equivalent if

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq X_n') < \infty$$

By the Borel-Cantelli Lemma (I), setting $A_n = \{X_n \neq X_n'\}$, we have $\mathbb{P}(A_n \text{ i.o.}) = 0$.

I.e. $\exists$ null set N s.t. $\forall \omega \in N^c$,
$X_n(\omega) = X_n'(\omega)$ $\forall$ but finitely many n.

Cor: If $\{X_n\}_{n=1}^{\infty}$, $\{X_n'\}_{n=1}^{\infty}$ are tail equivalent, and $b_n \uparrow \infty$, if $\exists$ r.v. X s.t.

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} X_j' = X \text{ a.s.}, \text{ then } \lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} X_j = X \text{ a.s.}$$

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=M}^{n} X_j \overset{\text{"}}{=} X_j' \quad \text{for any } M.$$

We'd like to find a sequence of cut-offs $X_n' = X_n \mathbb{1}_{|X_n| \le M_n}$ so that $\{X_n\}_{n=1}^{\infty}$, $\{X_n'\}_{n=1}^{\infty}$ are tail equivalent. To that end, we have:

Lemma: If $X \in L^1$ and $\varepsilon > 0$, then

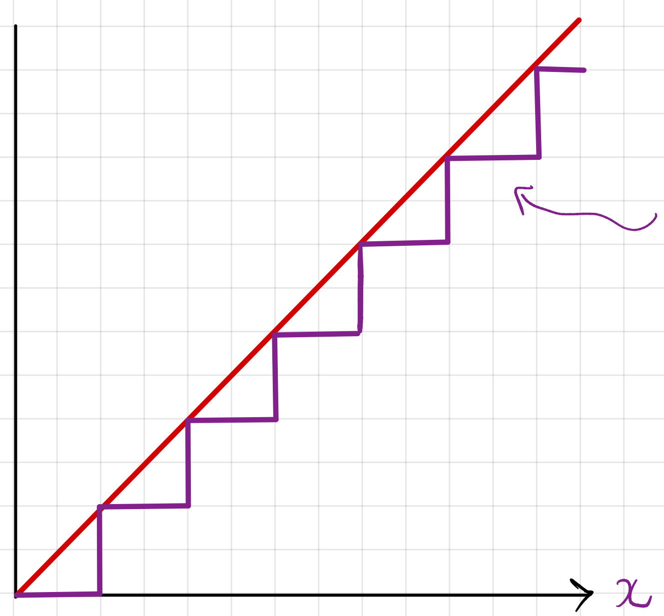
$$\sum_{n=1}^{\infty} \mathbb{P}(|X| \ge n\varepsilon) \le \frac{1}{\varepsilon} \mathbb{E}[|X|].$$

Markov:

$$\mathbb{P}(|X| \ge n\varepsilon) \le \frac{1}{n\varepsilon} \mathbb{E}[|X|]$$

$$\sum_{n=1}^{\infty} \left(\qquad \right) < \infty.$$

Pf.



$$x \mapsto \sum_{n=1}^{\infty} \mathbb{1}_{[n,\infty)}(x) \le x.$$

$$\therefore \underbrace{\mathbb{E}\left(\sum_{n=1}^{\infty} \mathbb{1}_{[n,\infty)}\left(\frac{|X|}{\varepsilon}\right)\right)}_{\mathbb{E}(\mathbb{1}_{\{|X| \ge n\varepsilon\}})} \le \mathbb{E}\left(\frac{|X|}{\varepsilon}\right) \qquad ///$$

Cor: If $\{X_n\}_{n=1}^{\infty}$ are (i)id and L^1, they are tail equivalent to $X_n' = X_n \mathbb{1}_{|X_n| \le n}$.

Pf. $$\sum_{n=1}^{\infty} \mathbb{P}(X_n' \ne X_n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_1| > n) \le \mathbb{E}[|X_1|] < \infty. \qquad ///$$

Thus, in order to prove the SLLN, it suffices to prove:

If $\{X_n\}_{n\geq 1}^{\infty}$ is an iid sequence of L^1 random variables with $\mathbb{E}[X_n]=\alpha$, and

$$S_n' = \sum_{k=1}^{n} X_k \mathbb{1}_{|X_k|\leq k},$$

then $\frac{S_n'}{n} \to \alpha$ a.s.

Advantages: bonded $\therefore L^2$

Disadvantages: X_n' not id. dst.