Summing Independent Random Variables

Let X, Y be two $\mathcal{N}(0,1)$ random variables.
What is the law of $X+Y$?

In general, this is an ill-posed question.

μ_{X+Y} is not determined by μ_X, μ_Y.

But it is determined by $\mu_{X,Y}$ - the joint law.

Let X, Y be Borel random vectors in $\mathbb{R}^d$. Then for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\mu_{X+Y}(B) = \int_B d\mu_{X+Y}$$

So μ_{X+Y} is completely determined by $\mu_{X,Y}$.

Now, if X, Y are independent,

Def. Given two Borel probability measures μ, ν on $\mathbb{R}^d$, their convolution $\mu * \nu$ is the probability measure on $\mathcal{B}(\mathbb{R}^d)$

$$\mu * \nu (B) := \int_{\mathbb{R}^{2d}} \mathbb{1}_B(x+y) \, \mu \otimes \nu (dx dy)$$

$$= \int_{\mathbb{R}^d} \nu(dx) \int_{\mathbb{R}^d} \mathbb{1}_B(x+y) \, \mu(dx)$$

As we saw (getting here), if we construct independent X, Y w/ $X \overset{d}{=} \mu$, $Y \overset{d}{=} \nu$,

$$\mu * \nu = \mathrm{Law}(X+Y)$$

$$\mu * \nu (B) = \int \mu(B-y)\, \nu(dy) = \int \nu(B-x)\, \mu(dx)$$

This is the best general formula for convolution.

If the measures are both $\ll \lambda^d$
or both pure point measures,
we can do better.

$$d\mu = f\, d\lambda^d \qquad f, g \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$$
$$d\nu = g\, d\lambda^d$$

$$\mu * \nu (B) = \int \mu(dx) \int \mathbb{1}_B(x+y)\, \nu(dy) = \int f(x)\, \lambda^d(dx) \int \mathbb{1}_B(x+y)\, g(y)\, \lambda^d(dy)$$

Prop: $\mu * \nu \ll \lambda^d$, and $\dfrac{d(\mu * \nu)}{d\lambda^d}$

In the pure point measure case:

$$\mu * \nu (B) = \int \nu(B-x)\, \mu(dx)$$

In particular, for any $u \in \mathbb{R}^d$,

$$\mu * \nu(\{u\}) = \int \nu(\{u-x\})\, \mu(dx)$$

E.g. $\mu = \text{Poiss}(\alpha)$, $\nu = \text{Poiss}(\beta)$.

$$\mu * \nu \{k\}$$

Eg. $\mu = \nu = \mathcal{N}(0,1)$

$$\frac{d\mu}{d\lambda}(x) = f_\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{d(\mu * \nu)}{d\lambda}(u) = f_\mu * f_\nu(u) = \int f_\mu(x) f_\nu(u-x)\,dx$$

A(n annoying) generalized version of this calculation shows

$$\mathcal{N}(\alpha, s^2) * \mathcal{N}(\beta, t^2) =$$

The Law of S_n

Let $\{X_n\}_{n=1}^{\infty}$ be iid random variables, with law μ. Let $S_n = X_1 + \cdots + X_n$.

Then $\mu_{S_n} =$

E.g. $\mu = \mathrm{Poiss}(\alpha)$. Then $\mu_{S_n} =$

$\mu = \mathcal{N}(0, \delta^2)$. Then $\mu_{S_n} =$

Rescaling: S_n / b_n for some $b_n \to \infty$.

$$\mathbb{E}\left[\frac{S_n}{b_n}\right] = \frac{1}{b_n}\mathbb{E}[S_n] = \frac{n}{b_n}\mathbb{E}[X_1]$$

$$\mathrm{Var}\left[\frac{S_n}{b_n}\right] = \frac{1}{b_n^2}\mathrm{Var}[S_n] = \frac{n}{b_n^2}\mathrm{Var}[X_1]$$

Embryonic Central Limit Theorem

$\{X_n\}_{n\geq 1}^{\infty}$ iid $\mathcal{N}(0,s^2)$ random variables, $S_n = X_1 + \cdots + X_n$.

Then for $b_n \in (0,\infty)$ $\qquad \mu_{S_n} = \mathcal{N}(0, ns^2)$

$\mathbb{P}(S_n/b_n \leq t)$

Theorem: Under the above conditions,

$$\mathrm{Law}(S_n/\sqrt{n})$$

More generally, if X_n are independent $\mathcal{N}(\alpha, s^2)$ random variables, then