Summing Independent Random Variables

Let X, Y be two $N(0,1)$ random variables.
What is the law of $X+Y$?

- Suppose $X=Y$. $\quad X+Y = 2X$

$$F_{2X}(t) = P(2X \le t) = P(X \le t/2) = F_X(t/2)$$

$$\therefore f_{2X}(t) = \frac{d}{dt} F_X(t/2) = \frac{1}{2} f_X(t/2) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t}{2}\right)^2} = \frac{1}{\sqrt{8\pi}} e^{-t^2/8}$$

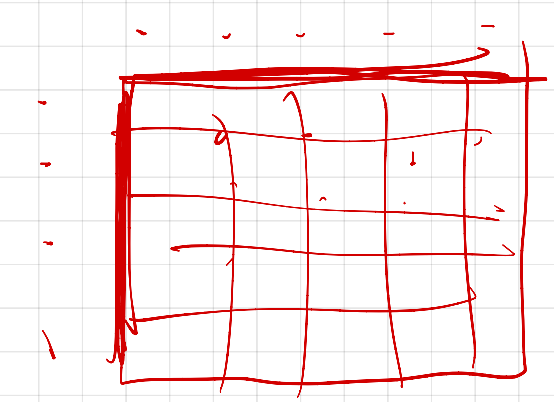
$X+Y \stackrel{d}{=} N(0,4)$. $\quad [\,\mathrm{Var}(2X) = 2^2\,\mathrm{Var}\,X\,]$

- Suppose $X = -Y$ $\quad$ if $Y \stackrel{d}{=} N(0,1)$ then $-Y \stackrel{d}{=} N(0,1)$ $\quad X+Y = 0$ $\quad X+Y \stackrel{d}{=} \delta_0$.

In general, this is an ill-posed question.

μ_{X+Y} is not determined by μ_X, μ_Y.

But it is determined by $\mu_{X,Y}$ – the joint law.



Let X, Y be Borel random vectors in $\mathbb{R}^d$. Then for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\mu_{X+Y}(B) = \int_B d\mu_{X+Y}$$

$$= \int_{\mathbb{R}^d} \mathbb{1}_B \, d\mu_{X+Y}$$

$$= E[\mathbb{1}_B(X+Y)] \qquad X+Y = f(X,Y)$$

$$= \int_{\mathbb{R}^{2d}} \mathbb{1}_B \circ f(x,y) \, \mu_{X,Y}(dx\,dy)$$

$$= \int_{\mathbb{R}^{2d}} \mathbb{1}_B(x+y) \, \mu_{X,Y}(dx\,dy)$$

So μ_{X+Y} is completely determined by $\mu_{X,Y}$.

Now, if X, Y are independent, $\mu_{X,Y} = \mu_X \otimes \mu_Y$

$$\mu_{X+Y}(B) = \int_{\mathbb{R}^{2d}} \mathbb{1}_B(x+y) \, \mu_X \otimes \mu_Y(dx\,dy)$$

&

$$= \int_{\mathbb{R}^d} \mu_Y(dy) \int_{\mathbb{R}^d} \mu_X(dx) \, \mathbb{1}_B(x+y).$$

Def. Given two Borel probability measures μ, ν on $\mathbb{R}^d$, their convolution $\mu * \nu$ is the probability measure on $\mathcal{B}(\mathbb{R}^d)$

$$\mu * \nu (B) := \int_{\mathbb{R}^{2d}} \mathbb{1}_B(x+y)\ \mu \otimes \nu (dx\,dy)$$

$$= \int_{\mathbb{R}^d} \nu(dx) \int_{\mathbb{R}^d} \underbrace{\mathbb{1}_B(x+y)}_{\substack{=1 \text{ iff } x+y \in B \\ \text{iff } x \in B-y.}} \mu(dx)$$

$$= \int_{\mathbb{R}^d} \nu(dx) \int_{\mathbb{R}^d} \mathbb{1}_{B-y}(x)\ \mu(dx)$$

$$= \int_{\mathbb{R}^d} \mu(B-y)\ \nu(dy)$$

As we saw (getting here), if we construct independent X, Y w $X \overset{d}{=} \mu$, $Y \overset{d}{=} \nu$,

$$\mu * \nu = \mathrm{Law}(X+Y) = \mathrm{Law}(Y+X) = \nu * \mu.$$

$$\mu * \nu (B) = \int \mu(B-y)\,\nu(dy) = \int \nu(B-x)\,\mu(dx)$$

This is the best general formula for convolution.

If the measures are both $\ll \lambda^d$ or both pure point measures, we can do better.

$d\mu = f\,d\lambda^d$
$d\nu = g\,d\lambda^d$ $\quad f, g \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$

$\int h(y+a)\,\lambda(dy)$
$= \int h(u)\,\lambda(du)$
b/c $\lambda(B+a) = \lambda(B)$

$$\mu * \nu (B) = \int \mu(dx) \int \mathbb{1}_B(x+y)\,\nu(dy) = \int f(x)\,\lambda^d(dx) \int \mathbb{1}_B(\underbrace{x+y}_{u})\,g(\underbrace{y}_{y = u-x})\,\lambda^d(dy)$$

$$= \int f(x)\,\lambda^d(dx) \int \mathbb{1}_B(u)\,g(u-x)\,\lambda^d(du)$$

$$= \int \mathbb{1}_B(u)\,\lambda^d(du) \underbrace{\int f(x)\,g(u-x)\,\lambda^d(dx)}_{f*g\,(u)}$$

$$= \int_B f*g\,d\lambda$$

Prop: $\mu * \nu \ll \lambda^d$, and $\dfrac{d(\mu*\nu)}{d\lambda^d} = f*g$

In the pure point measure case:

$$\mu * \nu (B) = \int \nu(B - x)\, \mu(dx)$$

In particular, for any $u \in \mathbb{R}^d$,

$$\mu * \nu(\{u\}) = \int \nu(\{u-x\})\, \mu(dx) = \sum_x \nu(\{u-x\})\, \mu(\{x\})$$

$$\sum_u \sum_x \nu(\{u-x\})\mu(\{x\}) = \sum_x \mu(\{x\}) \sum_u \nu(\{\underbrace{u-x}_{v \in \mathbb{R}^d}\})$$

$\therefore = 1.$ as $u \in \mathbb{R}^d$ also $u - x \in \mathbb{R}^d$

$$\sum_v \nu(\{v\}) = 1.$$

E.g. $\mu = \mathrm{Poiss}(\alpha)$, $\nu = \mathrm{Poiss}(\beta)$.

$$\mu * \nu \{k\} = \sum_x \mu(\{x\})\, \nu(\{k - x\})$$

$\mu(\{x\}) = 0$ unless $x \in \mathbb{N}$; $\nu(\{k-x\}) = 0$ unless $k - x \in \mathbb{N}$.

$\underbrace{\qquad}_{x \in \{0, 1, \dots, k\}}$

$$= \sum_{j=0}^{k} e^{-\alpha} \frac{\alpha^j}{j!} e^{-\beta} \frac{\beta^{k-j}}{(k-j)!}$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \alpha^j \beta^{k-j} = \frac{e^{-(\alpha+\beta)}}{k!} (\alpha+\beta)^k$$

$\therefore \mathrm{Poiss}(\alpha) * \mathrm{Poiss}(\beta) = \mathrm{Poiss}(\alpha+\beta)$

Eg. $\mu = \nu = \mathcal{N}(0,1)$

$$\frac{d\mu}{d\lambda}(x) = f_\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{d(\mu * \nu)}{d\lambda}(u) = f_\mu * f_\nu(u) = \int f_\mu(x) f_\nu(u-x)\,dx$$

$$= \frac{1}{2\pi} \int \underbrace{e^{-x^2/2} e^{-(u-x)^2/2}}_{\exp\left[\underbrace{-\frac{1}{2}x^2 - \frac{1}{2}(u-x)^2}_{-(x-u/2)^2 - \frac{1}{4}u^2}\right]} dx$$

μ_{X+Y} $\quad \mathcal{N}(0,1) \quad \mathcal{N}(0,1)$

$Var(X+Y) = Var X + Var Y = 1 + 1 = 2.$

$dx = dv$

$$= \frac{1}{2\pi} e^{-\frac{1}{4}u^2} \int_{\mathbb{R}} e^{-(\underbrace{x-u/2}_{v})^2} dx$$

$$= \frac{1}{2\pi} e^{-\frac{1}{4}u^2} \underbrace{\int e^{-v^2} dv}_{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{4\pi}} e^{-u^2/4}$$

density $\mathcal{N}(0,2)$

A (n annoying) generalized version of this calculation shows

$$\mathcal{N}(\alpha, s^2) * \mathcal{N}(\beta, t^2) = \mathcal{N}(\alpha+\beta, s^2+t^2).$$

The Law of S_n

Let $\{X_n\}_{n\geq 1}^{\infty}$ be iid random variables, with law μ. Let $S_n = X_1 + \cdots + X_n = S_{n-1} + X_n$

Then $\mu_{S_n} = \mu_{S_{n-1}} * \mu_{X_n}$

$$\mu_{X_1} * \mu_{X_2} * \cdots * \mu_{X_n}$$

E.g. $\mu = \text{Poiss}(\alpha)$. Then $\mu_{S_n} = \text{Poisson}(n\alpha)$ mean $= n\alpha$ Var $= n\alpha$

$\mu = N(0, \delta^2)$. Then $\mu_{S_n} = N(0, n\delta^2)$ mean $= 0$ Var $= n\delta^2$

Rescaling: S_n / b_n for some $b_n \to \infty$.

$$\mathbb{E}\left[\frac{S_n}{b_n}\right] = \frac{1}{b_n}\mathbb{E}[S_n] = \frac{n}{b_n}\mathbb{E}[X_1]$$ to stabilize mean, $b_n = n$.

$$\text{Var}\left[\frac{S_n}{b_n}\right] = \frac{1}{b_n^2}\text{Var}[S_n] = \frac{n}{b_n^2}\text{Var}[X_1]$$ to stabilize variance, $b_n = \sqrt{n}$.

Embryonic Central Limit Theorem

$\{X_n\}_{n\geq 1}^{\infty}$ iid $\mathcal{N}(0,s^2)$ random variables, $S_n = X_1 + \cdots + X_n$.

$\mu_{S_n} = \mathcal{N}(0, ns^2)$

Then for $b_n \in (0,\infty)$

$$\mathbb{P}(S_n/b_n \leq t) = \mathbb{P}(S_n \leq b_n t) = \int_{-\infty}^{b_n t} \frac{1}{\sqrt{2\pi n s^2}} e^{-x^2/2ns^2}\,dx$$

$x = b_n u$

$dx = b_n du$

$$= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi \frac{n}{b_n^2} s^2}} e^{-u^2/2\frac{n}{b_n^2}s^2}\,du = F_Z(t)$$

$Z \stackrel{d}{=} \mathcal{N}\left(0, \frac{n}{b_n^2}s^2\right)$

Theorem: Under the above conditions,

$$\mathrm{Law}(S_n/\sqrt{n}) = \mathcal{N}(0, s^2) \quad \forall n \in \mathbb{N}.$$

More generally, if X_n are independent $\mathcal{N}(\alpha, s^2)$ random variables, then

$$\mathrm{Law}\left(\frac{S_n - n\alpha}{\sqrt{n}\,s}\right) = \mathrm{Law}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}[S_n]}}\right) = \mathcal{N}(0,1) \quad \forall n$$