Tail Events

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The tail σ-field $\mathcal{T}$ of these rv's is

$$\mathcal{T} := \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, X_{n+2}, \dots)$$

Events $E \in \mathcal{T}$ are called tail events for the sequence $\{X_n\}_{n=1}^{\infty}$.

Eg. $\{\lim_{n\to\infty} X_n \text{ exists}\}$ & $\{\lim_{n\to\infty} X_n > 0\}$ $\in \mathcal{T}$

Eg. Let $S_n = X_1 + \cdots + X_n$.

$\{\lim_{n\to\infty} S_n \text{ exists}\} \in \mathcal{T}$

$\{\limsup_{n\to\infty} S_n > 0\}$

Be Careful !

Obviously, $\{X_1 < 1\} \notin T(X_n : n \in \mathbb{N})$ ... right?

Why? Because $\{X_1 < 1\} \notin \sigma(X_2, X_3, X_4, \dots)$ ... right?

Theorem. (Kolmogorov's 0-1 Law)
If $\{X_n\}_{n\geq 1}^{\infty}$ are independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then for any tail event $E \in \mathcal{T}(X_n : n \in \mathbb{N})$,

Pf. Let $\mathcal{B}_n = \sigma(X_1, \ldots, X_n)$.

By the "grouping" lemma,

$\mathcal{B}_n \uparrow$ as $n \uparrow \infty$, so $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is an algebra - in particular, a π-system.

Eg. The Borel - Cantelli lemma (I) + (II) together say: if $\{A_n\}_{n=1}^{\infty}$ are independent events

$$\mathbb{P}(A_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \end{cases}$$

Eg. Let $\{X_n\}_{n=1}^{\infty}$ be independent rv's.

Define $S_n = X_1 + \cdots + X_n$.

Let $b_n \in (0, \infty)$ s.t. $b_n \uparrow \infty$ as $n \uparrow \infty$.

$$\left\{ \lim_{n \to \infty} \frac{S_n}{b_n} = c \right\}$$

What kinds of random variables are τ-measurable?

To get some intuition, recall the Doob-Dynkin representation.

$$Y \text{ is } \sigma(X_1,\dots,X_n)\text{-measurable} \Rightarrow Y = F(X_1,\dots,X_n)$$

for a Borel function $F:\mathbb{R}^n \to \mathbb{R}$.

What about $\sigma(X_1, X_2, \dots)$-measurable functions?

Proposition: Let $\{X_n\}_{n=1}^{\infty}$ be random variables. Let $\varepsilon > 0$.
If Y is $\sigma(X_1, X_2, \dots)$-measurable and bounded, there is some $N \in \mathbb{N}$ and a Borel function $F:\mathbb{R}^N \to \mathbb{R}$ s.t.

$$\mathbb{E}[\,|Y - F(X_1,\dots,X_N)|\,] < \varepsilon.$$

So, if Y is $\mathcal{T} = \bigcap_{n\geq 1} \sigma(X_n, X_{n+1}, \ldots)$ - measurable,

it is $\sigma(X_n, X_{n+1}, \ldots)$-measurable $\forall n$, and $\therefore$

Y is "close" to a function of $X_n, X_{n+1}, \ldots$ $\forall n$.

This suggests that Y is a "function of nothing".

If $\{X_n\}_{n=1}^{\infty}$ are independent, this is rigorous.

Proposition: Let $\{X_n\}_{n=1}^{\infty}$ be independent. If Y is a $\overline{\mathbb{R}}$-valued random variable that is tail-measurable, then $\exists c \in \overline{\mathbb{R}}$ s.t. $Y = c$ a.s.

Pf.