Tail Events

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The tail σ-field $\mathcal{T}$ of these rv's is

$$\mathcal{T} := \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, X_{n+2}, \dots) = \mathcal{T}(X_n : n \in \mathbb{N})$$

Events $E \in \mathcal{T}$ are called tail events for the sequence $\{X_n\}_{n=1}^{\infty}$.

Eg. $\left\{ \lim_{n\to\infty} X_n \text{ exists} \right\}$ ✓ $\in \mathcal{T}$

& $\left\{ \lim_{n\to\infty} X_n > 0 \right\}$

$\left\{ \lim_{n\to\infty} X_n \text{ exists} \right\} = \left\{ \lim_{n\to\infty} X_n \mathbb{1}_{n \geq N} \text{ exists} \right\}$

$\in \sigma(X_N, X_{N+1}, \dots)$

Eg. Let $S_n = X_1 + \dots + X_n$.

$\left\{ \lim_{n\to\infty} S_n \text{ exists} \right\}$ ✓ $\in \mathcal{T}$

$\left\{ \underbrace{\limsup_{n\to\infty}}_{X_1} S_n > 0 \right\}$

Eg. $X_2, X_3, X_4, \dots \equiv 0$.

$\mathcal{T} \subseteq \sigma(X_2, X_3, X_4, \dots) = \{\emptyset, \Omega\}$

Be Careful !

Obviously, $\{X_1 < 1\} \notin \mathcal{T}(X_n : n \in \mathbb{N})$ ... right?

why? Because $\{X_1 < 1\} \notin \sigma(X_2, X_3, X_4, \dots)$ ... right?

What if $X_1 = X_2 = X_3 = \cdots$ $\quad \sigma(X_n, X_{n+1}, \dots) = \sigma(X_1)$

$\therefore = \mathcal{T}(X_n)$

$X_n = f_n(X_1) \qquad f_n^{-1}\mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}\sigma(X_n) = X_n^* \mathcal{B}(\mathbb{R}) &= \{X_n^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\} \\ &= \{X_1^{-1} \circ f_n^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\} \\ &= \sigma(X_1)\end{aligned}$$

Assume $X_n : n \in \mathbb{N}$ are independent.

Theorem. (Kolmogorov's 0-1 Law)

If $\{X_n\}_{n\geq 1}^{\infty}$ are independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then for any tail event $E \in \mathcal{T}(X_n : n \in \mathbb{N})$, $\mathbb{P}(E) = 0$ or 1.

$\mathcal{T}$ is "almost trivial"

Pf. Let $\mathcal{B}_n = \sigma(X_1, \dots, X_n) \overset{\text{independent}}{\longleftrightarrow} \sigma(X_{n+1}, X_{n+2}, \dots) \supseteq \mathcal{T}$

By the "grouping" lemma,

$\mathcal{B}_n$ and $\mathcal{T}$ are indep $\forall n$.

$\mathcal{B}_n \uparrow$ as $n \uparrow \infty$, so $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is an algebra - in particular, a π-system.

$\Rightarrow$ $\sigma(\mathcal{A})$ is indep from $\mathcal{T}$.

$\sigma(\mathcal{A}) = \sigma(X_1, X_2, X_3, \dots) \supseteq \mathcal{T}$.

$\mathbb{P}(E) = 0$ or 1.

$\forall E \in \mathcal{T}$, $\mathbb{P}(E) = \mathbb{P}(E \cap E) = \mathbb{P}(E)\mathbb{P}(E) = \mathbb{P}(E)^2$ ///

Eg. The Borel-Cantelli lemma (I)+(II) together say: if $\{A_n\}_{n=1}^{\infty}$ are independent events

$\Leftrightarrow \{\mathbb{1}_{A_n}\}_{n=1}^{\infty}$ are indep.

$$\mathbb{P}(A_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \end{cases}$$

$$\{A_n \text{ i.o.}\} = \underbrace{\bigcap_{n \geq 1} \bigcup_{k \geq n} A_k}$$

$\in \sigma(\mathbb{1}_{A_n}, \mathbb{1}_{A_{n+1}}, \ldots)$

$\in \mathcal{T}(\mathbb{1}_{A_n} : n \geq 1)$

Eg. Let $\{X_n\}_{n=1}^{\infty}$ be independent r.v's.

Define $S_n = X_1 + \cdots + X_n$.

Let $b_n \in (0, \infty)$ s.t. $b_n \uparrow \infty$ as $n \uparrow \infty$.

$$\left\{ \lim_{n \to \infty} \frac{S_n}{b_n} = c \right\} \in \mathcal{T}(X_n : n \geq 1)$$

$$\mathbb{P}\left(\frac{S_n}{b_n} \to c \right) = 0 \text{ or } 1.$$

$\Updownarrow$

$$\forall N \quad \lim_{n \to \infty} \frac{X_N + X_{N+1} + \cdots + X_n}{b_n} = c$$

$$\frac{S_n}{b_n} = \frac{X_1 + \cdots + X_{N-1} + X_N + X_{N+1} + \cdots + X_n}{b_n}$$

$$= \frac{S_{N-1}}{b_n} + \frac{X_N + X_{N+1} + \cdots + X_n}{b_n}$$

$\frac{S_{N-1}}{b_n} \to 0$

What kinds of random variables are $\mathcal{T}$-measurable?
To get some intuition, recall the Doob-Dynkin representation.

Y is $\sigma(X_1, \ldots, X_n)$-measurable $\Rightarrow$ $Y = F(X_1, \ldots, X_n)$
for a Borel function $F: \mathbb{R}^n \to \mathbb{R}$.

What about $\sigma(X_1, X_2, \ldots)$-measurable functions?

Proposition: Let $\{X_n\}_{n=1}^{\infty}$ be random variables. Let $\varepsilon > 0$.
If Y is $\sigma(X_1, X_2, \ldots)$-measurable and bounded, there is some $N \in \mathbb{N}$ and a Borel function $F: \mathbb{R}^N \to \mathbb{R}$ s.t.

$$\mathbb{E}[|Y - F(X_1, \ldots, X_N)|] < \varepsilon.$$

Let $\mathcal{A} = \bigcup_{n=1}^{\infty} \sigma(X_1, \ldots, X_n)$ $\qquad \sigma(\mathcal{A}) = \sigma(X_1, X_2, X_3, \ldots)$

HW: $\exists$ $\mathcal{A}$-simple φ s.t. $\mathbb{E}[|Y - \varphi|] < \varepsilon$.

$$\varphi = \sum_{k=1}^{m} \alpha_k \mathbb{1}_{A_k}, \quad A_k \in \mathcal{A}$$

$A_k \in \sigma(X_1, \ldots, X_{n_k}) \in \sigma(X_1, \ldots, X_N)$

$N = \max(n_1, \ldots, n_m)$

$$\Rightarrow \varphi = F(X_1, \ldots, X_N).$$

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So, if Y is $\mathcal{T} = \bigcap_{n\geq 1} \sigma(X_n, X_{n+1}, \ldots)$ - measurable,

it is $\sigma(X_n, X_{n+1}, \ldots)$-measurable $\forall n$, and $\therefore$

Y is "close" to a function of $X_n, X_{n+1}, \ldots$ $\forall n$.

This suggests that Y is a "function of nothing".

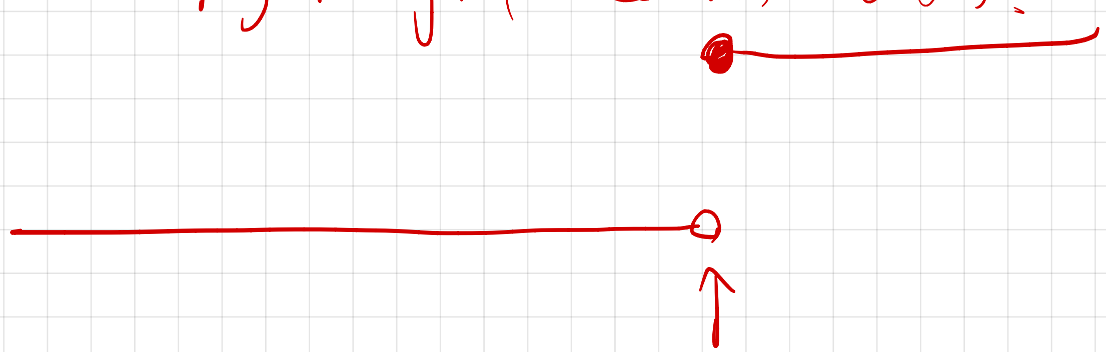
If $\{X_n\}_{n=1}^{\infty}$ are independent, this is rigorous.

Proposition: Let $\{X_n\}_{n=1}^{\infty}$ be independent. If Y is a $\overline{\mathbb{R}}$-valued random variable that is tail-measurable, then $\exists c \in \overline{\mathbb{R}}$ s.t. $Y = c$ a.s.

Pf. Assume $Y \in \mathbb{R}$ a.s. $F_Y(t) = \mathbb{P}(Y \leq t)$

$\{Y \leq t\} = Y^{-1}(-\infty, t] \in \mathcal{T}$.

$\uparrow$, right-continuous $\therefore$ $= 0$ or 1.



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